A Graphical Criterion for the Identification of Causal Effects in Linear Models

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Abstract

This paper concerns the assessment of direct causal effects from a combination of: (i) non-experimental data, and (ii) qualitative domain knowledge. Domain knowledge is encoded in the form of a directed acyclic graph (DAG), in which all interactions are assumed linear, and some variables are presumed to be unobserved. The paper establishes a sufficient criterion for the identifiability of all causal effects in such models as well as a procedure for estimating the causal effects from the observed covariance matrix.

Introduction

This paper explores the feasibility of inferring linear cause-effect relationships from various combinations of data and theoretical assumptions. The assumptions considered will be represented in the form of an acyclic causal diagram which contains both arrows and bi-directed arcs (Pearl 1995; 2000). The arrows represent the potential existence of direct causal relationships between the corresponding variables, and the bi-directed arcs represent spurious correlations due to unmeasured common causes. All interactions among variables are assumed to be linear. Our task will be to decide whether the assumptions represented in the diagram are sufficient for assessing the strength of causal effects from non-experimental data and, if sufficiency is proven, to express the target causal effect in terms of estimable quantities.

This decision problem has been tackled in the past half century, primarily by econometricians and social scientists, under the rubric "The Identification Problem" (Fisher 1966) – it is still unsolved. Certain restricted classes of models are nevertheless known to be identifiable, and these are often assumed by social scientists as a matter of convenience or convention [Wright, 1960; Duncan, 1975]. McDonald [1997] characterizes a hierarchy of three such classes: (1) no bidirected arcs, (2) bidirected arcs restricted to root variables, and (3) bidirected arcs restricted to variables that are not connected through directed paths. The structural equations in all three classes are regressional, and the parameters can therefore be estimated uniquely using OLS techniques (Bollen [1989, pp.104]).

Figure 1: (a) a "bow-pattern", and (b) a bow-free model

Recently, (Brito & Pearl 2002) have shown that the identification of the entire model is ensured if variables standing in direct causal relationship (i.e., variables connected by arrows in the diagram) do not have correlated errors; no restrictions need to be imposed on errors associated with indirect causes. This class of models was called "bow-free", since their associated causal diagrams are free of any "bow pattern" (Pearl 2000) (see Figure 1).

In this work, we provide a new sufficient graphical criterion for the identification of general linear models. The distinctive characteristic of our criterion is the fact that it does not rely on the conditional independences implied by the model. As a consequence, it can be successfully applied to prove the identification of models with few conditional independences, while most existing methods would fail.

The remainder of the paper is organized as follows. We begin with a brief introduction to linear models and the identification problem, and review some useful definitions. Then, we describe our approach and define the fundamental concept of Auxiliary Variable. Next, we give a complete characterization of the Auxiliary variables and present a sufficient graphical criterion for identification. Finally, we provide an algorithm to find a suitable set of auxiliary variables in the model.

Linear Models and Identification

An equation $Y = \beta X + e$ encodes two distinct assumptions: (1) the possible existence of (direct) causal influence of X on Y; and, (2) the absence of causal influence on Y of any variable that does not appear on the right-hand side of the equation. The parameter β quantifies the (direct) causal effect of X on Y. That is, the equation claims that a unit increase in X would result in β units increase of Y. The variable e is called an "error" or "disturbance"; it represents unobserved background factors that the modeler decides to keep unexplained.

A linear model for a set of random variables $\mathbf{Y}$ =

$$
\begin{aligned}
&Y_1 = e_1\\
&Y_2 = e_2\\
&Y_3 = aY_1 + e_3\\
&Y_4 = bY_2 + cY_3 + e_4\\
&Cov(e_1, e_2) = \alpha \neq 0\\
&Cov(e_2, e_3) = \beta \neq 0
\end{aligned}
$$

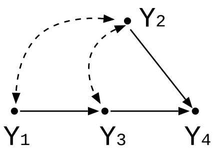

Figure 2: A simple linear model and its causal diagram

$\{Y_1, \ldots, Y_n\}$ is defined by a set of equations of the form

$$Y_j = \sum_i c_{ji} Y_i + e_j \qquad , j = 1, \ldots, n \tag{1}$$

where the error terms e_j are assumed to have normal distribution with zero mean, and variance/covariance matrix Ψ, $[\Psi_{ij}] = Cov(e_i, e_j)$.

The equations and the pairs of error-terms (e_i, e_j) with non-zero correlation define the structure of the model. The model structure can be represented by a directed graph, called causal diagram, in which the set of nodes is defined by the variables $Y_1, \ldots, Y_n$, and there is a directed edge from Y_i to Y_j if the coefficient of Y_i in the equation for Y_j is distinct from zero. Additionally, if error-terms e_i and e_j have non-zero correlation, we add a (dashed) bidirected edge between Y_i and Y_j. Figure 2 shows a simple model and the respective causal diagram.

The structural parameters of the model, denoted by θ, are the coefficients c_{ij}, and the values of the non-zero entries of the error covariance matrix Ψ. The models considered in this work are assumed to be recursive, that is, $c_{ji} = 0$ for $i \geq j$.

Fixing the model structure and assigning values to the parameters θ, the model determines a unique covariance matrix Σ over the observed variables $\{Y_1, \ldots, Y_n\}$, given by (see (Bollen 1989), page 85)

$$\Sigma(\theta) = (I - C)^{-1} \Psi \left[(I - C)^{-1} \right]' \tag{2}$$

where C is the matrix of coefficients c_{ji}.

Conversely, in the Identification problem, after fixing the structure of the model, one attempts to solve for θ in terms of the observed covariance Σ. This is not always possible. In some cases, no parametrization of the model could be compatible with a given Σ. In other cases, the structure of the model may permit several distinct solutions for the parameters. In these cases, the model is called *nonidentifiable*.

A convenient way of relating parameter identification to the structure of the model is to write Eq.(2) for each term σ_{ij} of Σ using Wright's method of path coefficients (Wright 1960). Wright's method consists of equating the (standardized) covariance σ_{ij} with the sum of products of parameters (or path coefficients) along unblocked paths between Y_i and Y_j (examples are given later). If the resulting equations give a unique solution to some path coefficient c_{ij}, independent of (unobserved) error correlations, that coefficient is identifiable.

Background

Definition 1 *A* path *in a graph is a sequence of edges (directed or bidirected) such that each edge starts in the node ending the preceding edge. A* directed path *is a path composed only by directed edges, all oriented in the same direction. We say that node X is an* ancestor *of node Y if there is a directed path from X to Y. A path is said to be* blocked *if there is a node Z and a pair of consecutive edges in the path such that both edges are oriented toward Z (e.g.,$\ldots \to Z \leftarrow \ldots$).*

Let p be a path between nodes X and Y. We say that path p points to X (Y) if the edge of p incident to X (Y) is oriented toward it. Let Z be an intermediate variable of path p. We denote the subpath of p consisting of the edges between X and Z by $p[X \sim Z]$.

Definition 2 *Define the* depth *of a node in a DAG as the length (in number of edges) of the longest directed path between the node and any of its ancestors. Nodes with no ancestors have depth 0.*

Definition 3 *Define the* distance *between two nodes X and Y, denote by $dist(X, Y)$, as the length (in number of edges) of the shortest path (blocked or unblocked) between variables X and Y, and for $\mathbf{Y} = \{Y_1, \ldots, Y_k\}$, we define $dist(X, \mathbf{Y}) = min_j\{dist(X, Y_j)\}$.*

Lemma 1 *Let X, Y be nodes in the causal diagram of a recursive model, such that depth(X) $\geq$ depth(Y). Then, every path between X and Y which includes a node Z with depth(Z) $\geq$ depth(X) must be blocked.*

Basic Approach

Our strategy for the identification problem is as follows. For a fixed variable Y, we assume that the parameters of edges connecting variables at depth smaller than Y are already identified. Then, we establish graphical conditions on the causal diagram such that the parameters of the edges incoming Y are identifiable. Once we have such a criterion, it is easy to implement an iterative procedure to verify the identifiability of the entire model by examining the variables in increasing depths.

Fix a variable Y in the model and let $depth(Y) = k$. Assume that the parameters of all edges connecting variables at depth smaller than k are identified. Let $\mathbf{X} = \{X_1, \ldots, X_m\}$ [1] be the set of variables at depth smaller than k which are connected to Y by an edge.

We further divide the variables in $\mathbf{X}$ into subsets $\overline{\mathbf{X}}$, $\widehat{\mathbf{X}}$ and $\widehat{\overline{\mathbf{X}}}$. A variable X_i belongs to $\overline{\mathbf{X}}$ if it is connected to Y only by a directed edge; $X_i \in \widehat{\mathbf{X}}$ if it is connected to Y only by a bidirected edge; and $X_i \in \widehat{\overline{\mathbf{X}}}$ if there is a directed and a bidirected edge between X_i and Y.

Define the following set of edges incoming Y:

$$Inc(Y) = \{(X_i, Y) : X_i \in \mathbf{X}\}$$

[1] In the following, we will also use letters X, Z, W to refer to variables in the model to avoid overloading the subscripts.

$\sigma_{24} = \beta\lambda_1 + a\alpha\lambda_1 + \lambda_2$
$\sigma_{34} = \lambda_1 + \beta\lambda_2 + a\alpha\lambda_2$

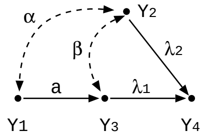

Figure 3: Wright's equations

Note that if $X_i \in \widehat{\overline{\mathbf{X}}}$, then there are two edges denoted by (X_i, Y) (one directed and one bidirected) and both of them are in $Inc(Y)$. So, if $|\widehat{\overline{\mathbf{X}}}| = n$, then

$$|Inc(Y)| = |\mathbf{X}| + |\widehat{\overline{\mathbf{X}}}| = m + n$$

For each $X_i \in \mathbf{X}$, we apply Wright's method to the pair $\{X_i, Y\}$, and obtain the following equations:

$$\sigma_{X_i,Y} = \sum_{\text{paths } p_l} T(p_l) \qquad ,i = 1, \ldots, m \qquad (3)$$

where term $T(p_l)$ is the product of the parameters of edges along the path p_l, and the summation is over all unblocked paths between X_i and Y. Figure 3 shows Eq. (3) for Y_4 as the fixed variable in a simple model.

In the following, we refer to the equation obtained by applying Wright's method to the pair $\{X_i, Y\}$ as the Wright's equation for X_i and Y.

Let $\lambda_1, \ldots, \lambda_{m+n}$, denote the parameters of the edges in $Inc(Y)$. Then, Eq. (3) can be rewritten as:

$$\sigma_{X_i,Y} = \sum_{j=1}^{m+n} a_{ij} \cdot \lambda_j \qquad ,i = 1, \ldots, m \qquad (4)$$

where terms in coefficient a_{ij} correspond to unblocked paths between X_i and Y including edge (X_j, Y).

Nonlinear terms (e.g., $\lambda_j\lambda_i$) do not appear in these equations, because each unblocked path from X_i to Y contains exactly one edge from $Inc(Y)$. Moreover, it follows from Lemma 1 and our assumptions, that all the factors appearing in a_{ij} are identified parameters.

Let Φ denote the system of linear equations (4). The following result was proved in (Brito & Pearl 2002):

Theorem 1 *The equations in Φ are linearly independent.*

If $\widehat{\overline{\mathbf{X}}} = \emptyset$, then Φ has m equations for m unknowns. Hence, theorem 1 guarantees that Φ has unique solution, and so the parameters $\lambda_1, \ldots, \lambda_m$ are identifiable

If $|\widehat{\overline{\mathbf{X}}}| = n > 0$, then we have to find n variables providing additional independent equations to obtain the identification of parameters λ_i's. This motivates the following definition:

Definition 4 *A variable Z at depth smaller than k is said to be an Auxiliary Variable if and only if the Wright's equation for Z and Y is linearly independent from the equations in Φ.*

We restrict ourselves to variables at depth smaller than k so that we maintain the desirable property that every factor appearing in the coefficients of the equation is an identified parameter.

The AV Criterion

Our definition of Auxiliary Variable is closely related to the well-known concept of Instrumental Variable, in the sense that both enable the identification of causal-effects. The traditional definition qualifies a variable Z as instrumental, relative to a cause X and effect Y if (Pearl 2000):

1. Z is independent of all error terms that have an influence on Y that is not mediated by X;
2. Z is not independent of X.

The intuition behind this definition is that all correlation between Z and Y must be intermediated by X. If we can find Z with these properties, then the causal effect of X on Y, denoted by c, is identified and given by $c = \sigma_{ZY}/\sigma_{ZX}$. In the following, we provide a weaker set of conditions that completely characterize the Auxiliary variables. The conditions are based on the existence of a path (or sequence of paths) between Z and X in the path diagram, with a few restrictions. No condition is imposed on the existence of alternative paths between Z and Y which do not go through X, except that Z cannot be connected to Y by an edge.

AV Criterion:

Variable Z satisfies the AV criterion if we can find $X_{i_1} \in \widehat{\overline{\mathbf{X}}}$ and $X_{i_2}, \ldots, X_{i_k} \in \overline{\mathbf{X}}$ such that:

(i) for $j = 1, \ldots, k-1$, there is an unblocked path p_j between X_{i_j} and $X_{i_{j+1}}$ pointing to both variables;

(ii) there is an unblocked path p_k between Z and X_{i_k} pointing to X_{i_k};

(iii) for $1 \leq j \leq k$, if some $X_l \in \mathbf{X}$ is an intermediate variable of path p_j, then we must have that $X_l \in \widehat{\mathbf{X}}$ and subpath $p_j[X_l \sim X_{i_{j+1}}]$ points to X_l (or subpath $p_k[X_l \sim Z]$, if $j = k$).

We call the sequence of paths $\langle p_1, \ldots, p_k \rangle$ an auxiliary chain C. The variables $Z, X_{i_1}, \ldots, X_{i_k}$ are the terminal variables of C, and any other variable appearing in some path p_j is called an intermediate variable of C.

Figure 4 shows some models in which the variable Z satisfies the AV criterion. For example, in model (a) we have a chain consisting of the edge $Z \rightarrow X_1$. In model (d), we have the chain with paths: $Z \rightarrow W_2 \rightarrow X_2$ and $X_2 \leftrightarrow W_2 \leftarrow W_1 \rightarrow X_1$. Figure 5 shows some models in which the variable Z does not satisfy the AV criterion. In each of those models, at least one condition of the AV criterion is not satisfied.

Theorem 2 *Let Z be such that $depth(Z) < depth(Y)$ and $Z \notin \mathbf{X}$. Then Z is an Auxiliary Variable if and only if Z satisfies the AV criterion.*

The GAV Criterion

As mentioned in section 3, if $|\widehat{\overline{\mathbf{X}}}| = n$, then we need n auxiliary variables to obtain the identification of the parameters λ_i's. Here, we provide a sufficient condition on a set of auxiliary variables $\mathbf{Z} = \{Z_1, \ldots, Z_n\}$ for the system consisting of Wright's equations for each variable in $\mathbf{X} \cup \mathbf{Z}$ and Y to be linearly independent.

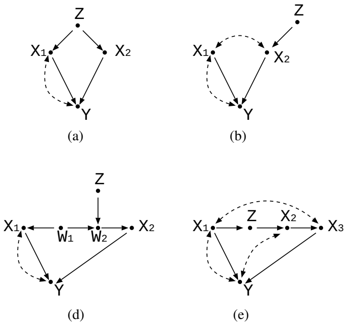

Figure 4: Variable Z satisfying AV criterion

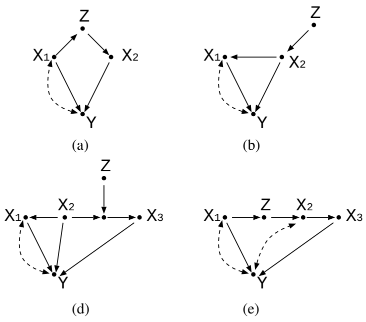

Figure 5: Variable Z not satisfying AV criterion

Intuitively, for each $X_i \in \widehat{\widehat{\mathbf{X}}}$ we should have a unique $Z_i \in \mathbf{Z}$ such that there is an auxiliary chain between Z_i and X_i. That is, two variables $X_i, X_j \in \widehat{\widehat{\mathbf{X}}}$ could not share the same auxiliary variable in $\mathbf{Z}$.

However, although necessary, this is not a sufficient condition. Next, we use the models in Figure 6 to illustrate the aspects in the structure of the model that allow to obtain independent equations.

In model (a), we have the chains $C_1 : \langle Z_1 \rightarrow Z_2 \rightarrow X_1 \rangle$, and $C_2 : \langle Z_2 \rightarrow X_2 \rangle$, but the system of equations provided by $\{X_1, X_2, Z_1, Z_2\}$ is not linearly independent. The problem seems to be that Z_2 appears in every chain between Z_1 and some $X_h \in \widehat{\widehat{\mathbf{X}}}$.

However, this is still not complete, since in model (b) Z_2 also appears in every chain between Z_1 and some $X_h \in \widehat{\widehat{\mathbf{X}}}$, and the equations provided by $\{X_1, X_2, X_3, Z_1, Z_2\}$ are linearly independent. Moreover, in model (d), neither Z_1 appears in any chain between Z_2 and some $X_h \in \widehat{\widehat{\mathbf{X}}}$, nor vice-versa, but a common variable W appears in every such chains, and as a result the equations provided by $\{X_1, X_2, Z_1, Z_2\}$ are not linearly independent.

Finally, observe that the orientation of the edges in the chains is an important issue. Note that the only difference between models (b) and (c) is the orientation of the edge (Z_2, X_1), but while the variables $\{X_1, X_2, X_3, Z_1, Z_2\}$ provide independent equations in model (b), this is not the case in model (c).

The GAV criterion presented below formalizes those ideas and provides a sufficient condition to obtain linearly independent equations.

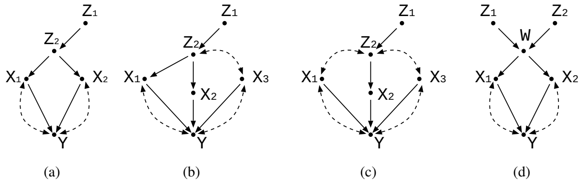

Figure 6: Models requiring 2 or more auxiliary variables

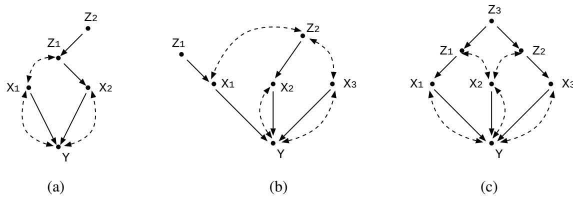

Figure 7: Variables satisfying the GAV criterion

The GAV Criterion:

Let $\widehat{\widehat{\mathbf{X}}} = \{X_1, \ldots, X_n\}$, and let $\mathbf{Z} = \{Z_1, \ldots, Z_n\}$ be a set of auxiliary variables. Then, $\mathbf{Z}$ satisfies the GAV criterion if and only if we can find auxiliary chains $C_1, \ldots, C_n$, such that

(i) C_i is a chain between Z_i and X_i;

(ii) no $X_j \in \mathbf{X}$ appears as a terminal variable in more than one chain;

(iii) if p_i and p_j are paths of distinct chains, then they do not have any intermediate variable in common;

(iv) if p is a path in chain C_j connecting the terminal variables X_l and X_{l+1}, and Z_i appears as intermediate variable in p, then both the subpath $p[Z_i \sim X_{l+1}]$ and the last path of chain C_i must point to variable Z_i.

Figure 7 shows some examples in which the variables Z_i marked in the model satisfy the GAV criterion. In model (a), we have chains $C_1 : \langle Z_1 \leftrightarrow X_1 \rangle$ and $C_2 \langle Z_2 \rightarrow Z_1 \rightarrow X_2 \rangle$, which clearly satisfy conditions $(i) - (iii)$ above. To see that condition (iv) also holds, note that both path $Z_1 \leftrightarrow X_1$ and subpath $Z_2 \rightarrow Z_1$ point to Z_1. We have a similar situation in model (b).

In model (c), we have more than one choice for the chains C_1, C_2 and C_3. If we take, for instance, $C_1 : \langle Z_1 \leftrightarrow X_2 \rangle$, $C_2 : \langle Z_2 \rightarrow X_3 \rangle$ and $C_3 : \langle Z_3 \rightarrow Z_1 \rightarrow X_1 \rangle$, then we see that they satisfy all the conditions $(i) - (iv)$.

Theorem 3 *If a set of Auxiliary variables $\mathbf{Z} = \{Z_1, \ldots, Z_n\}$ satisfies the GAV criterion, then the system consisting of Wright's equations for each variable in $\mathbf{X} \cup \mathbf{Z}$ and Y is linearly independent.*

The GAV criterion may appear somewhat restrictive at first. In fact, it is easy to find examples of sets of auxiliary variables which do not satisfy the GAV criterion, but still provide independent equations. However, in all such examples we could always find another set of auxiliary variables satisfying the GAV criterion. So, we conjecture that the GAV criterion is also a necessary condition to obtain independent equations from auxiliary variables.

Algorithm

In more elaborate models, it may not be an easy task to check if a set of auxiliary variables satisfies the GAV criterion. In this section we present an algorithm to find a set of auxiliary variables satisfying the GAV criterion, if such set exists. The basic idea is to reduce the problem to that of solving an instance of the maximum flow problem.

(Cormen, C.Leiserson, & Rivest 1990) define the maximum flow problem as follows. A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a non-negative capacity $c(u, v) \geq 0$. Two vertices are distinguished in the flow network: a source s and a sink t. A flow in G is a real-valued function $F : V \times V \to R$, satisfying:

- $F(u, v) \leq c(u, v)$, for all $u, v \in V$;
- $F(u, v) = -F(v, u)$, for all $u, v \in V$;
- $\sum_{v \in V} F(u, v) = 0$, for all $u \in V - \{s, t\}$.

The value of a flow F is defined as $|F| = \sum_{v \in V} F(s, v)$. In the maximum flow problem, we have to find a flow from s to t with maximum value.

Before describing the construction of the flow network G, we present an important result which allows to considerably reduce the number of candidates to compose the set of auxiliary variables.

Lemma 2 *If there is any set of auxiliary variables* $\mathbf{Z}'$ *satisfying the GAV criterion, then we can always find another set* $\mathbf{Z} = \{Z_1, \ldots, Z_n\}$ *which also satisfies the GAV criterion, such that:*

$$dist(Z_j, \mathbf{X}) \leq \lfloor log|\widehat{\widehat{\mathbf{X}}}| \rfloor + 1 \qquad , j = 1, \ldots, n$$

According to Lemma 2, we only need to consider variables with distance at most $\lfloor log|\widehat{\widehat{\mathbf{X}}}| \rfloor + 1$ from $\mathbf{X}$, to find a set of auxiliary variables satisfying the GAV criterion.

Now, the set of vertices of flow network G consists of:

- a vertex V_i for each variable $X_i \in \mathbf{X}$;
- vertices $V_{\widehat{Z}}$ and $V_{\overline{Z}}$, for each variable $Z \notin \mathbf{X}$, with $depth(Z) < depth(Y)$ and $dist(Z, \mathbf{X}) \leq \lfloor log|\widehat{\widehat{\mathbf{X}}}| \rfloor + 1$;
- a source vertex s, and a sink vertex t.

We have two vertices representing each variable $Z \notin \mathbf{X}$ because such variables may appear in more than one auxiliary chain (in fact, at most two).

The set E of edges in G is defined as follows. The goal is to have a correspondence between auxiliary chains in the model and directed paths from s to t in the flow network G. To obtain such correspondence, we include $V_i \to V_j$ in E if and only if the variables corresponding to V_i and V_j in the model, say X_i and X_j, are connected by an edge, and such edge can be traversed by a path in an auxiliary chain in the direction from X_i to X_j.

Let us analyze the situation for $X_i \in \overline{\mathbf{X}}$ and $X_j \in \widehat{\mathbf{X}}$. Assume that X_i and X_j are represented by vertices V_i and V_j, respectively, in G. If $X_i \to X_j$ is the only edge between X_i and X_j in the model, then we do not include any edge between V_i and V_j in E, because edge $X_i \to X_j$ cannot appear in a path of an auxiliary chain. If edge $X_j \to X_i$ is present in the model, then we include edge $V_j \to V_i$ in E but do not include $V_i \to V_j$, because edge $X_j \to X_i$ can only be traversed from X_j to X_i in a path of an auxiliary chain. Similarly, if edge $X_i \leftrightarrow X_j$ exists in the model, then we include edge $V_i \to V_j$ in the model but do not include $V_j \to V_i$.

In some cases, one edge in the model corresponds to two edges in E. For example, if $X_i, X_j \in \overline{\mathbf{X}}$ and we have edge $X_i \leftrightarrow X_j$ in the model, then we include edges $V_i \to V_j$ and $V_j \to V_i$ in E.

For edges incident to some $Z \notin \mathbf{X}$ we have a more complex procedure, because such variables are represented by two vertices in G, and we have to ensure that condition (iv) of the GAV criterion is satisfied. We omit the technical details here, and give a table with all types of edges in the model and the corresponding edges in G in the appendix.

The following edges are also required:

- for each $X_h \in \widehat{\widehat{\mathbf{X}}}$, we include $V_h \to t$;
- for each $Z \notin \mathbf{X}$, we include $s \to V_{\widehat{Z}}$;

Figure 9 shows an example of a model and the corresponding flow network.

In order to solve the maximum flow problem on G, we assign capacity 1 to every edge in E, and impose the additional constraint of maximum flow capacity of 1 to the vertices of G (this can be implemented by splitting each vertex into two and connecting them by an edge with capacity 1), except for vertices s and t.

We solve the maximum flow problem on G using the Ford-Fulkerson method and obtain a flow F. From the integrality theorem (see Cormen et al, p.603) we get that F allocates a non-negative integer amount of flow to each edge. Since we assign capacity 1 to every edge, we can interpret the solution F as a selection of directed paths $p_1, \ldots, p_{|F|}$, from s to t. Moreover, it follows from the additional constraint that these paths do not share any common vertex other than s and t.

Finally, note that vertex s is connected only to vertices $V_{\widehat{Z_i}}$, representing some $Z_i \notin \mathbf{X}$, and such vertices can appear only once in a directed path from s to t. Thus, each path p_i can be associated with a unique variable $Z_i \notin \mathbf{X}$. Hence, if $|F| = |\widehat{\widehat{\mathbf{X}}}|$, the algorithm returns the set of variables associated with paths $p_1, \ldots, p_{|F|}$. Figure 8 summarizes the steps of the algorithm.

Theorem 4 *The algorithm is sound, that is, the returned set of variables satisfies the GAV criterion.*

Theorem 5 *The algorithm is complete, that is, it always find a set of variables satisfying the GAV criterion if such set exists.*

Conclusion and Discussion

In this paper we have introduced a new graphical criterion for parameter identification in linear models. Most exist-

Algorithm:
1. Construct a flow network based on the model structure;
2. Solve the maximum flow problem on G using the Ford-Fulkerson method to obtain a flow F.
3. If $|F| = |\widehat{\overline{\mathbf{X}}}|$, return the set of variables associated with paths p_i. Otherwise, return the empty set.

Figure 8: Algorithm to find auxiliary variables.

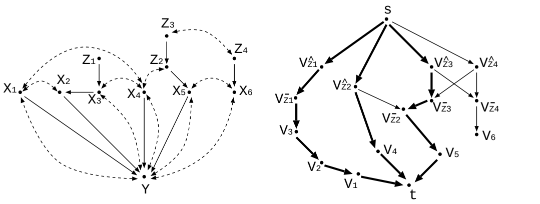

Figure 9: A model and the corresponding flow network

ing methods for this problem take advantage of the conditional independence relations implied by the model. Since our method does not make any use of this feature, it is appropriate even for models which are not rich in conditional independences.

Although our criterion can prove the identifiability of a large class of models, it is not complete. Figure 10 shows an example of a completely identifiable model for which our criterion fails. Manipulating the set of Wright's equations for each pair of variables in this model, one can prove the identification of every parameter.

Our criterion, fails because there is no variable at depth smaller than $depth(Y)$ satisfying the conditions of an Auxialiary variable. However, we note that variable Z has all properties to be an auxiliary variable, except that $depth(Z) > depth(Y)$. Thus, relaxing the definition of Auxiliary Variable to include such cases could be a natural extension of this work.

Appendix

Let $X_i, X_j \in \overline{\mathbf{X}}$, $X_l, X_k \in \overline{\mathbf{X}}$, $X_g, X_h \in \widehat{\overline{\mathbf{X}}}$, $Z, Z_1, Z_2 \notin \mathbf{X}$. Then, the correspondence of edges in the model and in the flow network is given by:

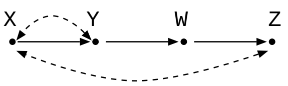

Figure 10: An identifiable model

$X_i \leftrightarrow X_j$	:	$V_i \rightarrow V_j$ and $V_j \rightarrow V_j$
$X_i \leftrightarrow X_l$	:	$V_i \rightarrow V_l$
$X_l \rightarrow X_i$	:	$V_l \rightarrow V_i$
$X_l \rightarrow X_k$	:	$V_l \rightarrow V_k$
$X_i \leftrightarrow X_h$	:	$V_i \rightarrow V_h$
$X_l \rightarrow X_h$	:	$V_l \rightarrow V_h$
$Z \rightarrow X_h$	:	$V_{\overline{Z}} \rightarrow V_h$
$Z \leftrightarrow X_h$	:	$V_{\widehat{Z}} \rightarrow V_h$
$Z \rightarrow X_l$	:	$V_{\overline{Z}} \rightarrow V_l$
$Z \leftrightarrow X_l$	:	$V_{\widehat{Z}} \rightarrow V_l$
$X_l \rightarrow Z$	:	$V_l \rightarrow V_{\overline{Z}}$
$Z \rightarrow X_i$	:	$V_{\overline{Z}} \rightarrow V_i$
$Z \leftrightarrow X_i$	:	$V_{\widehat{Z}} \rightarrow V_i$ and $V_i \rightarrow V_{\overline{Z}}$
$Z_1 \rightarrow Z_2$	:	$V_{\overline{Z}_1} \rightarrow V_{\overline{Z}_2}$
$Z_1 \leftrightarrow Z_2$	:	$V_{\widehat{Z}_1} \rightarrow V_{\overline{Z}_2}$ and $V_{\widehat{Z}_2} \rightarrow V_{\overline{Z}_1}$
$\forall Z \notin \mathbf{X}$	:	$V_{\widehat{Z}} \rightarrow V_{\overline{Z}}$

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