A General Methodology for Response and Delay Computations in VLSI Interconnects *

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Abstract

In this paper we obtain general expressions for voltage response and signal delay at any arbitrary location x of a distributed transmission line. We give a closed form expression for the exact time-domain response at any location x of a distributed RC line. We also obtain the Elmore delay at any arbitrary location x of a distributed line. Finally, using two-pole analysis we develop a New Delay model which incorporates inductance effects into the delay estimate.

1 Overview

This paper derives new expressions for response and delay at any arbitrary location x of a distributed line, using the $ABCD$ parameters. From the two-pole analysis, a New Delay model is developed, which is the first to incorporate inductance effects (e.g., Elmore delay is independent of inductance). The following lists our contributions (except where noted otherwise, all results are new in the literature).

- We give the exact time-domain response at any location x of an open-ended distributed RC line (Equation (7)). In order to compute delay estimates we approximate the time-domain response using single-pole and two-pole methodologies.
- Based on these results, we obtain various approximations for delay at any prescribed threshold voltage. Such delay models, e.g., those given in Equations (10) and (11), can be used to find the optimum location x on the line having a specified delay at a given threshold voltage.

*This work was supported by NSF Young Investigator Award MIP-9257982.

- We obtain the Elmore delay at any location x of a distributed RC line in Equation (8), matching the result obtained in the literature using Elmore delay for RC trees [RPH83, Wya87]. In Equation (20), we also give the Elmore delay at any location x in a distributed RLC line with resistive source and capacitive load. We also obtain the Elmore delay for a distributed $RLCG$ line in Equation (23).
- We obtain the transfer function, Elmore delay and single-pole delay estimates for a distributed RLC line at any location x with *general* source and *general* load impedances (Section 3). This allows us to correct the 90% threshold delay estimate obtained by Sakurai ([Sak83], improved in [Sak93]) for the end of a distributed RC line with resistive source and capacitive load (Equation (21)). Again, delays can be obtained at any given threshold. The accuracy of our approach can be easily extended based on the number of poles used to approximate the response.
- We develop a "New Delay" formula for distributed RLC lines that stems a the two-pole analysis. Unlike previous delay formulas (in particular, Elmore delay), the New Delay model incorporates the effect of inductance in the delay estimates. The New Delay model extends to a delay estimate for general interconnect trees.

The organization of the paper is as follows. In Section 2 we give an analysis which assumes an ideal source; we discuss the of the open-ended distributed RC and RLC lines, and the distributed RLC line with capacitive load, and for these cases we provide the response and delay estimates for any position x on the line. Section 3 extends the discussion to distributed RLC lines with general source and load impedance, while Section 4 gives the analysis for distributed $RLCG$ lines. Section 5 presents the New Delay model for distributed RLC lines and general interconnect trees.

2 Distributed Line with Ideal Source

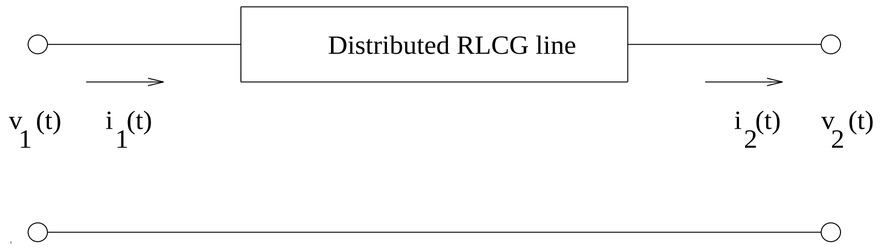

Figure 1: The 2-port model of a distributed $RLCG$ line.

The $ABCD$ parameters of a distributed $RLCG$ transmission line (Figure 1) are [Dwo79]:

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta h) & Z_0 \sinh(\theta h) \\ \frac{1}{Z_0}\sinh(\theta h) & \cosh(\theta h) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix} \tag{1}$$

where $\theta = \sqrt{(r+sl)(g+sc)}$; r, l, c, g are resistance, inductance, capacitance, conductance *per unit length* and h is the length of the line. $V_1(s), I_1(s)$ and $V_2(s), I_2(s)$ correspond to the voltage and current at, respectively, the left (input) port and right (output) port of the interconnect line. $Z_0 = \sqrt{\frac{R+sL}{G+sC}}$ is the characteristic impedance of the distributed line, where R, L, C, G are the total resistance, inductance, capacitance and conductance of the distributed line of length h.

The voltage and current at any location x of the distributed line, in terms of the input (left-port) voltage and current, are:

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta x) & Z_0 \sinh(\theta x) \\ \frac{1}{Z_0}\sinh(\theta x) & \cosh(\theta x) \end{pmatrix} \begin{pmatrix} V_x(s) \\ I_x(s) \end{pmatrix}$$

The voltage and current at any location x in terms of the output (right-port) voltage and current are given without derivation in [GK68, Sak83] (see Appendix A for a derivation):

$$\begin{pmatrix} V_x(s) \\ I_x(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta(h-x)) & Z_0 \sinh(\theta(h-x)) \\ \frac{1}{Z_0}\sinh(\theta(h-x)) & \cosh(\theta(h-x)) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix} \tag{2}$$

From Equations (1) and (2), the voltage at the beginning of the line and at any x can be expressed in terms of the voltage and current at the end of the line as

$$\begin{aligned} V_1(s) &= V_2(s)\cosh(\theta h) + I_2(s) Z_0 \sinh(\theta h) \\ V_x(s) &= V_2(s)\cosh(\theta(h-x)) + I_2(s) Z_0 \sinh(\theta(h-x)) \end{aligned}$$

Let Z_T be the load at the end of the distributed line. Substituting $V_2(s) = Z_T I_2(s)$ and dividing, we obtain the transfer function $H(x,s)$ between the point x and the left (input) port of the line.

$$H(x,s) = \frac{V_x(s)}{V_1(s)} = \frac{Z_T \cosh(\theta(h-x)) + Z_0 \sinh(\theta(h-x))}{Z_T \cosh(\theta h) + Z_0 \sinh(\theta h)} \tag{3}$$

Equation (3) is a key step: it allows us to calculate the voltage response for any value of the load impedance (Z_T), e.g., for open-ended, short-circuited, etc. cases. We now analyze the response at any position x on the open-ended RC line, the distributed RLC line with capacitive load, and the distributed $RLCG$ line.

2.1 Open-ended Distributed RC Line

A case of special interest in the literature is the open-ended line, for which $Z_T \to \infty$ ($I_2(s) = 0$). For this case, Equation (3) reduces to

$$H(x,s) = \frac{\cosh(\theta(h-x))}{\cosh(\theta h)} \tag{4}$$

For a step input, $V_1(s) = \frac{V_0}{s}$, where V_0 is the magnitude of the step input. Therefore, the voltage response at any x is given by

$$V_x(s) = \frac{V_0}{s} \cdot \frac{\cosh(\theta(h-x))}{\cosh(\theta h)} \tag{5}$$

Substituting $\theta(h-x) = \sqrt{sRC(1-\frac{x}{h})^2}$ and $Z_0 = \sqrt{\frac{R}{sC}}$ and expanding the cosh terms in Equation (4) into infinite series, the transfer function at any x of the distributed RC line is

$$H(x,s) = \frac{1 + \frac{sRC}{2!}(1-\frac{x}{h})^2 + \frac{(sRC)^2}{4!}(1-\frac{x}{h})^4 + ...}{1 + \frac{sRC}{2!} + \frac{(sRC)^2}{4!} + ...} \tag{6}$$

The exact time-domain response at any x can be calculated from Equation (5) (see Appendix B) and is given by

$$\begin{aligned} v_x(t) \quad = \quad & V_0 \left\{ erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) + \sum_{n=1}^{\infty}(-1)^n erfc\left(\sqrt{\frac{R_{2nh+x}C_{2nh+x}}{4t}}\right) \right. \\ & \left. + \sum_{n=1}^{\infty}(-1)^{n-1} erfc\left(\sqrt{\frac{R_{2nh-x}C_{2nh-x}}{4t}}\right)\right\} \end{aligned} \tag{7}$$

where $R_{2nh\pm x} = (2nh \pm x)r$ and $C_{2nh\pm x} = (2nh \pm x)c$. To obtain efficient delay estimates, various approximations of the response are of interest.

2.1.1 Single-Pole Analysis

Elmore Delay

Elmore delay [Elm48] is defined to be the first moment of the system impulse response, i.e., the coefficient of s in the system transfer function $H(s)$ (refer to Appendix C for background). The output response in the Laplace domain for an impulse input is $H(s)$, and the impulse response in the time domain is $h(t)$. Therefore, Elmore delay is $T_{ED} = \int_0^\infty th(t)dt$. A detailed discussion of equivalent Elmore delay definitions is given in Appendix E. Although here we are interested only in computing the rise time delays, effects of time-of-flight on propagation delay are discussed in Appendix F.

Applying the Elmore definition to the transfer function $H(x,s)$ in Equation (6), the delay at any position x is

$$\begin{aligned} T_{ED}(x) &= \frac{RC}{2} - \frac{RC}{2}(1-\frac{x}{h})^2 \\ &= \frac{RC}{2}\left(1-(1-\frac{x}{h})^2\right) \\ &= \frac{RC}{2}(\frac{2x}{h}-\frac{x^2}{h^2}) \\ &= \frac{rc}{2}(2xh-x^2) \end{aligned} \tag{8}$$

As a special case, the delay at the end of the line, i.e. at $x = h$, is $T_{ED}(h) = \frac{RC}{2}$. Wyatt [Wya87] gives a similar expression for Elmore delay at any x by considering the portion of the line between x and the end of the line as the capacitive load to the line at x, then using the integral definition of Elmore delay (see Appendix E).

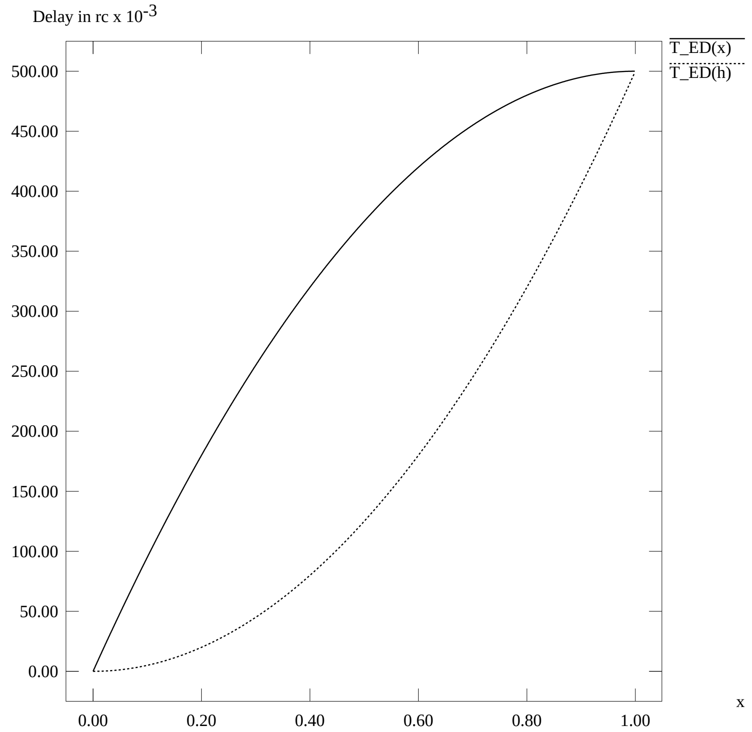

Figure 2: The Elmore delay at any position x, $T_{ED}(x)$ and the approximated Elmore delay at any x, $T_{approx}(x)$ are plotted. The x-axis indicates the position x as a fraction of total length of the line h and the y-axis indicates the delay interms of rc.

In current practice, the delay at any position x on the distributed RC line is calculated by assuming that the line is of length x, and then computing delay at the *end* of the line, i.e., $T_{approx}(x) = \frac{rc}{2}(x^2)$; see

[WE92] for an example. Recent works in the physical design area of VLSI CAD have used this expression to motivate "cost-radius" tradeoffs for performance-driven routing tree construction. However, Equation (8) significantly affects the construction of performance-driven routing trees, with respect to both the routing topology and the location of Steiner points. The Figure **??** shows the difference between the Elmore delay in Equation (8) and the previous approximated Elmore delay at x, $T_{approx}(x) = \frac{rc}{2}(x^2)$. In particular, when $x \ll h$ the Elmore delay $T_{ED}(x) = rcxh$ is linearly proportional to x.

Delay at Arbitrary Threshold Voltage

The voltage response at any x can be approximated from Equation (5) using a single-pole analysis. (Single-pole and two-pole analyses are also discussed in [Hor84].) Here, we develop three different single-pole approximations.

Approximation 1:

$$\begin{aligned} V_x(s) &= \frac{V_0}{s} \cdot \frac{1 + \frac{sRC}{2}(1-\frac{x}{h})^2 + \ldots}{1 + \frac{sRC}{2} + \ldots} = \frac{V_0}{s} \cdot \frac{1}{(1 + \frac{sRC}{2} + \ldots)(1 + \frac{sRC}{2}(1-\frac{x}{h})^2 + \ldots)^{-1}} \\ &\approx \frac{V_0}{s} \cdot \frac{1}{1 + s(\frac{RC}{2} - \frac{RC(1-\frac{x}{h})^2}{2})} = \frac{V_0}{s} \cdot \frac{1}{1 + sT_{ED}(x)} \\ &\approx V_0 \left(\frac{1}{s} - \frac{1}{s + \frac{1}{T_{ED}(x)}} \right) \end{aligned}$$

Taking the inverse Laplace transform, we obtain the time-domain response

$$v_x(t) \approx V_0(1 - e^{-\frac{t}{T_{ED}(x)}}) \tag{9}$$

Approximation 2:

$$V_x(s) \approx \frac{V_0}{s} \cdot \frac{\cosh(\sqrt{sRC(1-\frac{x}{h})^2})}{1 + \frac{sRC}{2}} \approx V_0 \left(\frac{1}{s} - \frac{\cosh(\sqrt{-2(1-\frac{x}{h})^2})}{s + \frac{1}{RC/2}} \right)$$

and again taking the inverse Laplace transform yields the time-domain response

$$v_x(t) \approx V_0 \left[1 - \cos\left(\sqrt{2(1-\frac{x}{h})^2} \right) e^{-\frac{t}{RC/2}} \right] \tag{10}$$

Approximation 3:

A variation of **Approximation 2** involves expanding the cosine term in Equation (10) and approximating to only the first few terms of the series.

$$\begin{aligned} v_x(t) &\approx V_0 \left[1 - \cos\left(\sqrt{2(1-\frac{x}{h})^2}\right) e^{-\frac{t}{RC/2}}\right] \approx V_0 \left[1 - \left(1 - (1-\frac{x}{h})^2\right) e^{-\frac{t}{RC/2}}\right] \\ &\approx V_0 \left[1 - (\frac{2x}{h} - \frac{x^2}{h^2}) e^{-\frac{t}{RC/2}}\right] \end{aligned} \quad (11)$$

The corresponding delays for any threshold $0 < Thr < 1$, i.e., as a fraction of V_0, are:

$$\begin{aligned} T_{Thr}(x) &= \ln\left(\tfrac{1}{1-Thr}\right) \cdot \tfrac{RC}{2}\left(1 - (1-\tfrac{x}{h})^2\right) = \ln\left(\tfrac{1}{1-Thr}\right) \cdot T_{ED}(x) && \text{for Approximation 1} \\ T_{Thr}(x) &= T_{ED}(h) \ln\left((\tfrac{1}{1-Thr}) \cdot \cos\left(\sqrt{2}(1-\tfrac{x}{h})\right)\right) && \text{for Approximation 2} \\ T_{Thr}(x) &= T_{ED}(h) \ln\left((\tfrac{1}{1-Thr}) \cdot \tfrac{T_{ED}(x)}{T_{ED}(h)}\right) && \text{for Approximation 3} \end{aligned}$$

For example, the 90% threshold delays at any x for the three approximations are:

$$\begin{aligned} T_{0.9}(x) &= 2.3\tfrac{RC}{2}\left(1 - (1-\tfrac{x}{h})^2\right) = 2.3 T_{ED}(x) && \text{for Approximation 1} \\ T_{0.9}(x) &= T_{ED}(h) \ln\left(10 \cos\left(\sqrt{2}(1-\tfrac{x}{h})\right)\right) && \text{for Approximation 2} \\ T_{0.9}(x) &= T_{ED}(h) \ln\left(10 \tfrac{T_{ED}(x)}{T_{ED}(h)}\right) && \text{for Approximation 3} \end{aligned}$$

Equation (11) is notable because it can also be used to determine the proper value for x which achieves a specified delay at a given threshold voltage, i.e.,

$$(1 - \frac{x}{h}) = \sqrt{\left(1 - (1 - V_{Thr}) e^{\frac{T_{Thr}}{T_{ED}(h)}}\right)}$$

where V_{Thr} is the threshold voltage and T_{Thr} is the corresponding delay at that threshold. Such a delay approximation is useful for interconnect design with prescribed delays, prescribed-skew, min-max delay constraints, etc.

2.1.2 Two-Pole Analysis

More accurate approximations for the voltage response are obtained using the two-pole methodology discussed in [Hor84, ZSTGC92, KM93]. The calculation of the time-domain voltage response in the standard two-pole analysis using the first and second moments of the transfer function is reviewed in Appendix D. From Equation (6), the first and second moments (see Appendix C for definitions) are given by

$$\begin{aligned} M_1 &= b_1 - a_1 = \frac{rc}{2}(2xh - x^2) \\ M_2 &= a_2 - a_1 b_1 + b_1^2 - b_2 = \frac{(RC)^2}{24}\left((1-\frac{x}{h})^4 - 1\right) + \frac{(RC)^2}{4}\left((1-\frac{x}{h})^2 - 1\right) \end{aligned}$$

Depending on the sign of $(4M_2 - 3M_1^2)$ the poles of the two-pole response will be either real or complex and the corresponding time-domain response can be obtained (see Appendix D).

Since the standard two-pole methodology in Appendix D needs only the first and second moments of the transfer function, it is effectively uses only up to the coefficient of s^2 in the numerator polynomial. Here, we give an alternate two-pole approximation which uses the *exact* numerator polynomial and approximates the denominator polynomial to two poles. The two-pole response obtained by approximating the denominator polynomial of Equation (5) to second degree is

$$V_x(s) = \frac{V_0}{s} \cdot \frac{\cosh(\sqrt{sRC(1-\frac{x}{h})^2})}{1 + s\frac{RC}{2} + s^2\frac{(RC)^2}{24}} = \frac{V_0}{s} \cdot \frac{\cosh(\sqrt{sRC(1-\frac{x}{h})^2})}{\frac{(RC)^2}{24}(s-s_1)(s-s_2)}$$

where $s_{1,2} = \frac{1}{RC}(-6 \pm \sqrt{12})$. By partial fractions, we obtain

$$V_x(s) = V_0\left(\frac{1}{s} - \frac{\sqrt{12}\cos(\sqrt{(6-\sqrt{12})(1-\frac{x}{h})^2})}{(6-\sqrt{12})}\frac{1}{s-s_1} + \frac{\sqrt{12}\cos(\sqrt{(6+\sqrt{12})(1-\frac{x}{h})^2})}{(6+\sqrt{12})}\frac{1}{s-s_2}\right)$$

The inverse Laplace transform yields the time-domain voltage response

$$v_x(t) = V_0\left(1 - \frac{\sqrt{12}\cos(\sqrt{(6-\sqrt{12})(1-\frac{x}{h})^2})}{(6-\sqrt{12})}e^{-t\frac{(6-\sqrt{12})}{RC}} + \frac{\sqrt{12}\cos(\sqrt{(6+\sqrt{12})(1-\frac{x}{h})^2})}{(6+\sqrt{12})}e^{-t\frac{(6+\sqrt{12})}{RC}}\right)$$

We may easily extend these single-pole and two-pole analyses to other cases of the distributed RLC line.

2.2 Open-ended Distributed RLC Line

Substituting $\theta(h-x) = \sqrt{(R+sL)sC(1-\frac{x}{h})^2}$ and $Z_0 = \sqrt{\frac{R+sL}{sC}}$ for a distributed RLC line, the open-ended transfer function in Equation (4) becomes

$$\begin{aligned} H(x,s) &= \frac{\cosh(\theta(h-x))}{\cosh(\theta h)} = \frac{1 + \frac{sC(R+sL)(1-\frac{x}{h})^2}{2!} + \frac{s^2C^2(R+sL)^2(1-\frac{x}{h})^4}{4!} + \ldots}{1 + \frac{sC(R+sL)}{2!} + \frac{s^2C^2(R+sL)^2}{4!} + \ldots} \\ &= \frac{1 + s\frac{RC}{2}(1-\frac{x}{h})^2 + s^2(\frac{(RC)^2}{24}(1-\frac{x}{h})^4 + \frac{LC}{2}(1-\frac{x}{h})^2) + \ldots}{1 + s\frac{RC}{2} + s^2(\frac{(RC)^2}{24} + \frac{LC}{2}) + \ldots} \end{aligned} \tag{12}$$

2.2.1 Single-pole Analysis

The Elmore delay for the open-ended distributed RLC line is given by

$$T_{ED}(x) = \frac{RC}{2} - \frac{RC}{2}(1-\frac{x}{h})^2 = \frac{rc}{2}(2xh - x^2)$$

This is the same as the Elmore delay for a distributed RC line in Equation (8), since inductance does not affect the first moment of the transfer function. The single-pole analysis of distributed RLC line is also similar to that of the distributed RC line: using **Approximation 1**, the threshold delay is same as that of distributed RC line, $T_{Thr}(x) = \ln\left(\frac{1}{1-Thr}\right) \cdot T_{ED}(x)$; using **Approximation 2** the threshold delay is

$$T_{Thr}(x) = T_{ED}(h)\ln\left((\frac{1}{1-Thr})\cos\left((1-\frac{x}{h})\sqrt{(2-\frac{4L}{R^2C})}\right)\right);$$

and **Approximation 3** implies that the value of x for a given delay time to a given threshold delay is

$$(1-\frac{x}{h}) = \sqrt{\frac{\left(1-(1-V_{Thr})e^{\frac{T_{Thr}}{T_{ED}(h)}}\right)}{1-\frac{2L}{R^2C}}}$$

2.2.2 Two-pole Analysis

The time-domain two-pole response is obtained from the first and second moments of the transfer function in Equation (12),

$$\begin{aligned} M_1 &= \frac{RC}{2} - \frac{RC}{2}(1-\frac{x}{h})^2 \\ M_2 &= \frac{(RC)^2}{24}\left((1-\frac{x}{h})^4 - 1\right) + \frac{LC}{2}\left((1-\frac{x}{h})^2 - 1\right) + \frac{(RC)^2}{4}\left(1-(1-\frac{x}{h})^2\right) \end{aligned}$$

For step input V_0, if we approximate the denominator of the transfer function up to the coefficient of s^2 then the two-pole response is

$$\begin{aligned} V_x(s) &\approx \frac{V_0}{s} \cdot \frac{\cosh(\sqrt{sC(R+sL)(1-\frac{x}{h})^2})}{1 + s\frac{RC}{2} + s^2(\frac{(RC)^2}{24} + \frac{LC}{2})} \\ &\approx \frac{V_0}{s} \cdot \frac{\cosh(\sqrt{sC(R+sL)(1-\frac{x}{h})^2})}{(\frac{(RC)^2}{24} + \frac{LC}{2})(s-s_1)(s-s_2)} \end{aligned}$$

where

$$s_{1,2} = \frac{-\frac{RC}{2} \pm \sqrt{\frac{(RC^2)}{12} - 2LC}}{\frac{(RC)^2}{12} + LC}$$

Using partial fractions,

$$
\begin{aligned}
V_x(s) &= V_0 \left(\frac{1}{s} + \frac{(\frac{(RC)^2}{12} + LC)\cosh(\sqrt{s_1 C(R+s_1 L)(1-\frac{x}{h})^2})}{(\sqrt{\frac{(RC^2)}{12} - 2LC})(-\frac{RC}{2} + \sqrt{\frac{(RC^2)}{12} - 2LC})} \cdot \frac{1}{s-s_1} \right. \\
&\quad \left. + \frac{(\frac{(RC)^2}{12} + LC)\cosh(\sqrt{s_2 C(R+s_2 L)(1-\frac{x}{h})^2})}{(\sqrt{\frac{(RC^2)}{12} - 2LC})(\frac{RC}{2} + \sqrt{\frac{(RC^2)}{12} - 2LC})} \cdot \frac{1}{s-s_2} \right) \\
&= V_0 \left(\frac{1}{s} - \frac{(\frac{(RC)^2}{12} + LC)\cosh(\sqrt{s_1 C(R+s_1 L)(1-\frac{x}{h})^2})}{(\sqrt{\frac{(RC^2)}{12} - 2LC})(\frac{RC}{2} - \sqrt{\frac{(RC^2)}{12} - 2LC})} \cdot \frac{1}{s-s_1} \right. \\
&\quad \left. + \frac{(\frac{(RC)^2}{12} + LC)\cosh(\sqrt{s_2 C(R+s_2 L)(1-\frac{x}{h})^2})}{(\sqrt{\frac{(RC^2)}{12} - 2LC})(\frac{RC}{2} + \sqrt{\frac{(RC^2)}{12} - 2LC})} \cdot \frac{1}{s-s_2} \right)
\end{aligned}
$$

Therefore, the new approximated time-domain response is

$$
\begin{aligned}
v_x(t) &= V_0 \left(\frac{1}{s} - \frac{(\frac{(RC)^2}{12} + LC)\cosh(\sqrt{s_1 C(R+s_1 L)(1-\frac{x}{h})^2})}{(\sqrt{\frac{(RC^2)}{12} - 2LC})(\frac{RC}{2} - \sqrt{\frac{(RC^2)}{12} - 2LC})} \cdot e^{ts_1} \right. \\
&\quad \left. + \frac{(\frac{(RC)^2}{12} + LC)\cosh(\sqrt{s_2 C(R+s_2 L)(1-\frac{x}{h})^2})}{(\sqrt{\frac{(RC^2)}{12} - 2LC})(\frac{RC}{2} + \sqrt{\frac{(RC^2)}{12} - 2LC})} \cdot e^{ts_2} \right)
\end{aligned}
$$

2.3 Distributed RLC Line With Capacitive Load

From Equation (3) the transfer function for a distributed RLC line with capacitive load $Z_T = \frac{1}{sC_T}$ is given by

$$
\begin{aligned}
H(x,s) &= \frac{\cosh(\theta(h-x)) + \frac{Z_0}{Z_T}\sinh(\theta(h-x))}{\cosh(\theta h) + \frac{Z_0}{Z_T}\sinh(\theta h)} \\
&= \frac{\left(1 + \frac{sC(R+sL)}{2!}(1-\frac{x}{h})^2 + \ldots\right) + (sC_T\sqrt{\frac{R+sL}{sC}})\left(\frac{\sqrt{sC(R+sL)(1-\frac{x}{h})^2}}{1!} + \frac{(sC(R+sL)(1-\frac{x}{h})^2)^{\frac{3}{2}}}{3!} + \ldots\right)}{\left(1 + \frac{sC(R+sL)}{2!} + \frac{(sC(R+sL))^2}{4!} + \ldots\right) + (sC_T\sqrt{\frac{R+sL}{sC}})\left(\frac{\sqrt{sC(R+sL)}}{1!} + \frac{(sC(R+sL))^{\frac{3}{2}}}{3!} + \ldots\right)} \\
&= \frac{1 + a_1 s + a_2 s^2 + \ldots}{1 + b_1 s + b_2 s^2 + \ldots} \qquad (13)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{RC(1-\frac{x}{h})^2}{2} + RC_T(1-\frac{x}{h}) \\
a_2 &= \frac{(RC)^2(1-\frac{x}{h})^4}{24} + \frac{LC(1-\frac{x}{h})^2}{2} + \frac{R^2CC_T(1-\frac{x}{h})^3}{6} + LC_T(1-\frac{x}{h}) \\
b_1 &= \frac{RC}{2} + RC_T
\end{aligned}
$$

$$b_2 = \frac{(RC)^2}{24} + \frac{LC}{2} + \frac{R^2CC_T}{6} + LC_T$$

The Elmore delay for this case is

$$\begin{aligned} T_{ED}(x) &= \frac{RC}{2} + RC_T - \frac{RC}{2}(1-\frac{x}{h})^2 - RC_T(1-\frac{x}{h}) \\ &= r\left(\frac{c}{2}(2xh - x^2) - xC_T\right) \end{aligned} \quad (14)$$

The response using single-pole analysis can be calculated as above. With **Approximation 1**, the threshold delay is $T_{Thr}(x) = \ln\left(\frac{1}{1-Thr}\right) \cdot T_{ED}(x)$; **Approximation 2** gives

$$T_{Thr}(x) = T_{ED}(h)\ln\left((\frac{1}{1-Thr})\cos\left((1-\frac{x}{h})\sqrt{\frac{RC}{b_1} - \frac{LC}{b_1^2}}\right) + \frac{C_T}{b_1}\sqrt{\frac{L-Rb_1}{C}}\sinh\left((1-\frac{x}{h})\sqrt{\frac{RC}{b_1} - \frac{LC}{b_1^2}}\right)\right)$$

The two-pole response can also be calculated as above; the response in transform domain for a step input is given by

$$V_x(s) = \frac{V_0}{s} \cdot \frac{\cosh\left((1-\frac{x}{h})\sqrt{(R+sL)sC}\right) + sC_T\sqrt{\frac{R+sL}{sC}}\sinh\left((1-\frac{x}{h})\sqrt{(R+sL)sC}\right)}{1 + s(\frac{RC}{2} + RC_T) + s^2(\frac{(RC)^2}{24} + \frac{LC}{2} + \frac{(R^2CC_T)}{6} + LC_T) + \ldots}$$

As described above, the time-domain response can be again obtained using partial fractions and taking inverse Laplace transforms.

The transfer function, single-pole response and two-pole response for a distributed RC line with capacitive load are obtained by substituting $L = 0$ in the distributed RLC results. Again, Elmore delay is identical for the distributed RC and distributed RLC cases.

3 Distributed Line with General Source Impedance

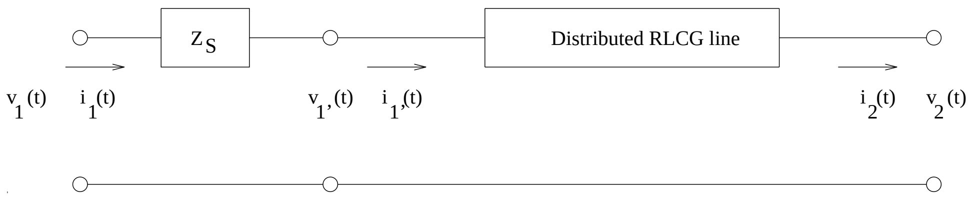

Figure 3: 2-port model of a distributed $RLCG$ line with source impedance Z_S.

We now develop delay and response computations for the distributed line with general source impedance Z_S in series. The more general case of series and parallel source impedance is discussed in Appendix G. The $ABCD$ parameters for the source impedance (Figure 3) are

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} 1 & Z_S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_{1'}(s) \\ I_{1'}(s) \end{pmatrix} \quad (15)$$

Substituting Equation (1) into Equation (15) and multiplying the matrices yields

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta h) + \frac{Z_S}{Z_0} Z_S \sinh(\theta h) & Z_0 \sinh(\theta h) + Z_S \cosh(\theta h) \\ \frac{1}{Z_0} \sinh(\theta h) & \cosh(\theta h) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix} \tag{16}$$

Therefore, the voltage at the beginning of the line is

$$V_1(s) = V_2(s) \left[\cosh(\theta h) + \frac{Z_S}{Z_0} \sinh(\theta h) \right] + I_2(s) \left[Z_0 \sinh(\theta h) + Z_S \cosh(\theta h) \right]$$

From the Equation (2), the voltage at any x is given by

$$V_x(s) = V_2(s) \cosh(\theta(h - x)) + I_2(s) Z_0 \sinh(\theta(h - x))$$

The transfer function at an x is obtained by substituting $V_2(s) = Z_T I_2(s)$ in the above equations and dividing them,

$$H(x,s) = \frac{V_x(s)}{V_1(s)} = \frac{\cosh(\theta(h-x)) + \frac{Z_0}{Z_T} \sinh(\theta(h-x))}{\left[\cosh(\theta h) + \frac{Z_S}{Z_0} \sinh(\theta h) \right] + \frac{1}{Z_T} \left[Z_0 \sinh(\theta h) + Z_S \cosh(\theta h) \right]} \tag{17}$$

Since the techniques are familiar, we give only two examples of the analysis; the second is of interest since it may be used to correct a widely-used estimate due to Sakurai [Sak83, Sak93].

3.1 Open-ended distributed RLC line

For an open-ended line ($I_2(s) = 0$) with source resistance $Z_S = R_S$ the transfer function reduces to

$$\begin{aligned} H(x,s) &= \frac{\cosh\left(\sqrt{sC(R+sL)(1-\frac{x}{h})^2}\right)}{\cosh\left(\sqrt{sC(R+sL)}\right) + R_S \sqrt{\frac{sC}{R+sL}} \sinh\left(\sqrt{sC(R+sL)}\right)} \\ &= \frac{1 + s\frac{RC(1-\frac{x}{h})^2}{2} + s^2 \left(\frac{LC(1-\frac{x}{h})^2}{2} + \frac{(RC)^2(1-\frac{x}{h})^4}{24} \right) + \ldots}{\left(1 + s\frac{RC}{2} + s^2(\frac{LC}{2} + \frac{(RC)^2}{24}) + \ldots\right) + \left(sR_SC + s^2 \frac{R_SRC^2}{6} + \ldots\right)} \\ &= \frac{1 + s\frac{RC(1-\frac{x}{h})^2}{2} + s^2 \left(\frac{LC(1-\frac{x}{h})^2}{2} + \frac{(RC)^2(1-\frac{x}{h})^4}{24} \right) + \ldots}{1 + s\left(R_SC + \frac{RC}{2}\right) + s^2 \left(\frac{LC}{2} + \frac{(RC)^2}{24} + \frac{R_SRC^2}{6} \right) + \ldots} \end{aligned} \tag{18}$$

Thus, Elmore delay at any x for a distributed RLC line with resistive source impedance is

$$T_{ED}(x) = b_1 - a_1 = R_SC + \frac{RC}{2} - \frac{RC(1-\frac{x}{h})^2}{2} = \frac{rc}{2}(2xh - h^2) + R_Sch$$

The response for single-pole analysis can be calculated using the approaches described earlier. The threshold delay using **Approximation 1** is $T_{Thr}(x) = \ln\left(\frac{1}{1-Thr}\right) \cdot T_{ED}(x)$, and **Approximation 2** gives

$$T_{Thr}(x) = T_{ED}(h) \ln\left((\frac{1}{1-Thr}) \cdot \cos\left((1 - \frac{x}{h}) \sqrt{-(\frac{RC}{b_1} - \frac{LC}{b_1^2})} \right) \right).$$

The time-domain response using two-pole analysis follows from the first and second moments of the Equation (18) and the methodology described in Appendix D.

3.2 Distributed RC line with capacitive load

Finally, we conclude this section by giving response and delay computations for the special case of a distributed RC line with resistive source and capacitive load. Our single-pole analysis is of interest since it can be compared with a similar result by Sakurai [Sak83]. For the distributed RC line with resistive source $Z_S = R_S$ and capacitive load $Z_T = \frac{1}{sC_T}$, the transfer function reduces to

$$H(x,s) = \frac{\left(1 + \frac{sRC}{2!}(1-\frac{x}{h})^2 + \frac{(sRC)^2}{4!}(1-\frac{x}{h})^4 + \ldots\right) + sC_T\sqrt{\frac{R}{sC}}\left(\sqrt{sRC(1-\frac{x}{h})^2} + \frac{1}{3!}(sRC(1-\frac{x}{h})^2)^{\frac{3}{2}} + ..\right)}{1 + s\left(\frac{RC}{2} + R_SC_T + RC(\frac{R_S}{R} + \frac{C_T}{C})\right) + s^2\left(\frac{(RC)^2}{24} + \frac{RCR_SC_T}{2} + \frac{(RC)^2}{6}(\frac{R_S}{R} + \frac{C_T}{C})\right) + \ldots} \tag{19}$$

Elmore's definition implies

$$T_{ED}(x) = \frac{RC}{2} + R_SC_T + RC(\frac{R_S}{R} + \frac{C_T}{C}) - \frac{RC}{2}(1-\frac{x}{h})^2 - RC_T(1-\frac{x}{h}) \tag{20}$$

The threshold delay using **Approximation 1** is $T_{Thr}(x) = \ln\left(\frac{1}{1-Thr}\right) \cdot T_{ED}(x)$, and in particular the delay at 90% threshold voltage is $T_{0.9} = 2.3T_{ED}(x)$. The delay at the end of the line can then be calculated by substituting $x = h$ in the above equations:

$$\begin{aligned} T_{ED}(h) &= \frac{RC}{2} + R_SC_T + R_SC + RC_T \\ T_{0.9}(h) &= 1.15RC + 2.3\left(R_SC_T + RC(\frac{R_S}{R} + \frac{C_T}{C})\right) \end{aligned} \tag{21}$$

Sakurai [Sak83] has also given the response and delay calculations for distributed RC line. He calculates the time-domain response from the transfer function using the Heaviside expansion over poles of the transfer function. Then, to calculate the delay he approximates the response using a single pole and observes the variation of delay with respect to source and load parameters; the 90% threshold delay is *heuristically* obtained as

$$T_{0.9}(h) = 1.02RC + 2.21\left(R_SC_T + RC(\frac{R_S}{R} + \frac{C_T}{C})\right) \qquad \text{in Sakurai, 1983}$$

This expression is almost same as the delay obtained by our single-pole analysis. In [Sak93] Sakurai corrected his 90% delay estimate, and the result comes closer to our single-pole result (note the new constant in the second term):

$$T_{0.9}(h) = 1.02RC + 2.3\left(R_SC_T + RC(\frac{R_S}{R} + \frac{C_T}{C})\right) \qquad \text{in Sakurai, 1993}$$

In the two-pole analysis, note that the first and second moments of the transfer function in Equation (19) are given by

$$
\begin{aligned}
M_1 &= T_{ED}(x) = \frac{RC}{2} + R_S C_T + RC(\frac{R_S}{R} + \frac{C_T}{C}) - \frac{RC}{2}(1-\frac{x}{h})^2 - RC_T(1-\frac{x}{h}) \\
M_2 &= \Bigg\{ \left(\frac{(RC)^2}{24}(1-\frac{x}{h})^4 + \frac{R^2 C C_T}{6}(1-\frac{x}{h})^3 \right) \\
&- \left(\frac{RC}{2}(1-\frac{x}{h})^2 + RC_T(1-\frac{x}{h}) \right) \cdot \left(\frac{RC}{2} + R_S C_T + RC(\frac{R_S}{R} + \frac{C_T}{C}) \right) \\
&+ \left(\frac{RC}{2} + R_S C_T + RC(\frac{R_S}{R} + \frac{C_T}{C}) \right)^2 - \left(\frac{(RC)^2}{24} + \frac{RC R_S C_T}{2} + \frac{(RC)^2}{6}(\frac{R_S}{R} + \frac{C_T}{C}) \right) \Bigg\} \quad (22)
\end{aligned}
$$

Using the previous methodology, the two-pole response for a step input is then given by

$$
V_x(s) = \frac{\cosh\left(\sqrt{sRC(1-\frac{x}{h})^2}\right) + sC_T\sqrt{\frac{R}{sC}}\sinh\left(\sqrt{sRC(1-\frac{x}{h})^2}\right)}{1 + s\left(\frac{RC}{2} + R_S C_T + RC(\frac{R_S}{R} + \frac{C_T}{C})\right) + s^2\left(\frac{(RC)^2}{24} + \frac{RC R_S C_T}{2} + \frac{(RC)^2}{6}(\frac{R_S}{R} + \frac{C_T}{C})\right) + \ldots}
$$

In this case, the poles of the transfer function are real because

$$
\left(\frac{RC}{2} + R_S C_T + RC(\frac{R_S}{R} + \frac{C_T}{C}) \right)^2 - \left(\frac{(RC)^2}{24} + \frac{RC R_S C_T}{2} + \frac{(RC)^2}{6}(\frac{R_S}{R} + \frac{C_T}{C}) \right) > 0
$$

As described in previous sections, the time-domain response can be obtained by applying partial fractions and inverse Laplace transforms.

4 Distributed $RLCG$ Line

In this section, we briefly discuss the case of a distributed $RLCG$ line with resistive source and capacitive load. From Equation (17), the transfer function at any x on the distributed $RLCG$ line is

$$
H(x,s) = \frac{\cosh(\theta(h-x)) + sC_T Z_0 \sinh(\theta(h-x))}{\left[(1 + sR_S C_T)\cosh(\theta h) + (\frac{R_S}{Z_0} + sC_T Z_0)\sinh(\theta h)\right]}
$$

Substituting for θ, Z_0 and expanding the cosh and sinh as infinite series yields the coefficients of the transfer function, as summarized in Appendix H. In contrast to the previous cases, here the coefficients of the s terms of the transfer function are themselves infinite series. Thus, we approximate the coefficients up to second degree terms of R, L, C and first degree terms of G (usually the conductance G is much smaller than the other parameters), yielding

$$
a_1 = \left(\frac{(RC + LG)(1-\frac{x}{h})^2}{2} + \frac{R^2 CG(1-\frac{x}{h})^4}{12} \right) + \left(RC_T(1-\frac{x}{h}) + \frac{R^2 C_T G(1-\frac{x}{h})^3}{6} \right)
$$

$$
\begin{aligned}
a_2 &= \left(\frac{LC(1-\frac{x}{h})^2}{2} + \frac{(R^2C^2+4RLCG)(1-\frac{x}{h})^4}{24}\right) + \left(LC_T(1-\frac{x}{h}) + \frac{2RLC_TG + R^2CC_T(1-\frac{x}{h})^3}{6}\right) \\
b_1 &= R_SC_T\left(1+\frac{RG}{2}\right) + \left(\frac{RC+LG}{2} + \frac{R^2CG}{12})\right) + \left(RC_T + \frac{R^2C_TG}{6}\right) + \left(R_SC + \frac{R_SRCG}{3}\right) \\
b_2 &= \left(\frac{LC}{2} + \frac{R^2C^2+4RLCG}{24}\right) + R_SC_T\left(\frac{RC+LG}{2} + \frac{2R^2CG}{24}\right) + \left(LC_T + \frac{2RLC_TG + R^2CC_T}{6}\right) \\
&\quad + \left(\frac{R_SRC^2 + 2R_SLCG}{6}\right)
\end{aligned}
$$

The Elmore delay at position x is

$$
\begin{aligned}
T_{ED}(x) &= b_1 - a_1 \\
&= R_SC_T\left(1+\frac{RG}{2}\right) + \left(\frac{RC+LG}{2} + \frac{R^2CG}{12})\right) + \left(RC_T + \frac{R^2C_TG}{6}\right) + \left(R_SC + \frac{R_SRCG}{3}\right) \\
&\quad - \left(\frac{(RC+LG)(1-\frac{x}{h})^2}{2} + \frac{(2RLG^2+2R^2CG)(1-\frac{x}{h})^4}{24}\right) + \left(RC_T(1-\frac{x}{h}) + \frac{R^2C_TG(1-\frac{x}{h})^3}{6}\right)
\end{aligned}
$$

and for the end of the line ($x = h$) we have

$$
T_{ED}(h) = b_1 = R_SC_T\left(1+\frac{RG}{2}\right) + \left(\frac{RC+LG}{2} + \frac{R^2CG}{12})\right) + \left(RC_T + \frac{R^2C_TG}{6}\right) + \left(R_SC + \frac{R_SRCG}{3}\right) \tag{23}
$$

Note that this reduces to Equations (20) and (21) when $G = 0$. The single-pole and two-pole responses can be approximated using the approaches given above.

5 The New Delay Model

The Elmore delay model is identical for RC and RLC distributed networks since inductance does not affect the first moment of the transfer function. Nevertheless, the Elmore model is widely used in delay estimation of routing trees and in the design of routing trees. This section presents a "New Delay" model which considers the effect of inductance. We will give example computations of 90% threshold delay estimates; the method trivially extends to delay estimates at any other threshold. To the best of our knowledge, this is the first delay model for arbitrary threshold voltages which incorporates the effects of inductance of a distributed line. There are many practical implications and research directions which stem from this new model, as summarized in the conclusions.

5.1 Open-ended Distributed RLC Line

For an open-ended distributed RLC line, recall that the transfer function at the end of the line is

$$H(s) = \frac{1}{\cosh(\theta h)} = \frac{1}{1 + s\frac{RC}{2} + s^2\left(\frac{(RC)^2}{24} + \frac{LC}{2}\right) + \ldots}$$

Here, $b_1 = \frac{RC}{2}$ and $b_2 = \frac{(RC)^2}{24} + \frac{LC}{2}$, implying $M_1 = b_1 = \frac{RC}{2}$ and $M_2 = b_1^2 - b_2 = \frac{(RC)^2}{12} - \frac{LC}{2}$. The two-pole methodology yields the following response for the case of real poles:

$$v(t) = V_0(1 - \frac{s_2}{s_2 - s_1}e^{s_1 t} + \frac{s_1}{s_2 - s_1}e^{s_2 t})$$

where

$$s_{1,2} = \frac{2}{-M_1 \pm \sqrt{4M_2 - 3M_1^2}} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2b_2}.$$

Real Poles:

The condition for the poles to be real is $\sqrt{4M_2 - 3M_1^2} = \sqrt{b_1^2 - 4b_2} \geq 0$. This is equivalent to a condition on the inductance of the line, namely

$$L \leq \frac{R^2C}{24} \qquad \text{i.e.,} \qquad l \leq \frac{h^2r^2c}{24}$$

Note that it is an $O(1)$ operation to determine whether the poles are real, complex, or double from the parameters of the line. Since $s_2 - s_1 = -\frac{\sqrt{b_1^2 - 4b_2}}{b_2}$ is negative, the coefficients $\frac{s_2}{s_2 - s_1}$ and $\frac{s_1}{s_2 - s_1}$ are positive. Also, since the magnitude $|s_2|$ is much greater than $|s_1|$, the second term in the time-domain response decreases rapidly compared to the first term. Hence, the two-pole response can be approximated (lower-bounded) as

$$v(t) \approx V_0(1 - \frac{s_2}{s_2 - s_1}e^{s_1 t}) \tag{24}$$

The approximation in Equation (24) is the key step toward the New Delay model. Since the voltage is lower bounded, the delay obtained is an upper bound on the actual delay. The delay at, e.g., $0.9V_0$ threshold can be estimated as

$$s_1\tau_{0.9} = \ln\left(\frac{s_2 - s_1}{10s_2}\right) = -ln\left(5[1 + \frac{b_1}{\sqrt{b_1^2 - 4b_2}}]\right)$$

Let

$$K_r = ln\left(5[1 + \frac{b_1}{\sqrt{b_1^2 - 4b_2}}]\right) = \left(5[1 + \frac{1}{\sqrt{1 - \frac{4b_2}{b_1^2}}}]\right)$$

We show that this K_r (the subscript r indicates real poles) can be determined by reconciling with a known delay estimate for distributed RLC lines. That is to say,

$$\begin{aligned}\tau_{0.9} &= \frac{-K_r}{s_1} = \frac{K_r}{|s_1|} = K_r \frac{-M_1 \pm \sqrt{4M_2 - 3M_1^2}}{2} = K_r \frac{2b_2}{b_1 - \sqrt{b_1^2 - 4b_2}} \\ &= K_r \frac{\frac{(RC)^2}{12} + LC}{\frac{RC}{2} - \sqrt{\frac{(RC)^2}{12} - 2LC}} \qquad (25)\end{aligned}$$

and substituting $L = 0$ gives a delay estimate for the distributed RC line $\tau_{0.9} = K_r \frac{RC}{(6-\sqrt{12})}$. But we know from the distributed RC analysis that $\tau_{0.9} = 2.3\tau_{ED} = 1.15RC$, or from the exact analysis of [Wil71] we have $\tau_{0.9} = 1.03RC$. Using the latter value, we find $K_r \approx 2.61$, i.e.,

$$\tau_{0.9} = 2.61 \frac{\frac{(RC)^2}{12} + LC}{\frac{RC}{2} - \sqrt{\frac{(RC)^2}{12} - 2LC}}$$

The condition on inductance for the poles to be real depends on the length h of the line. For present interconnect technologies [BKR93], generally $L \leq \frac{R^2C}{24}$.

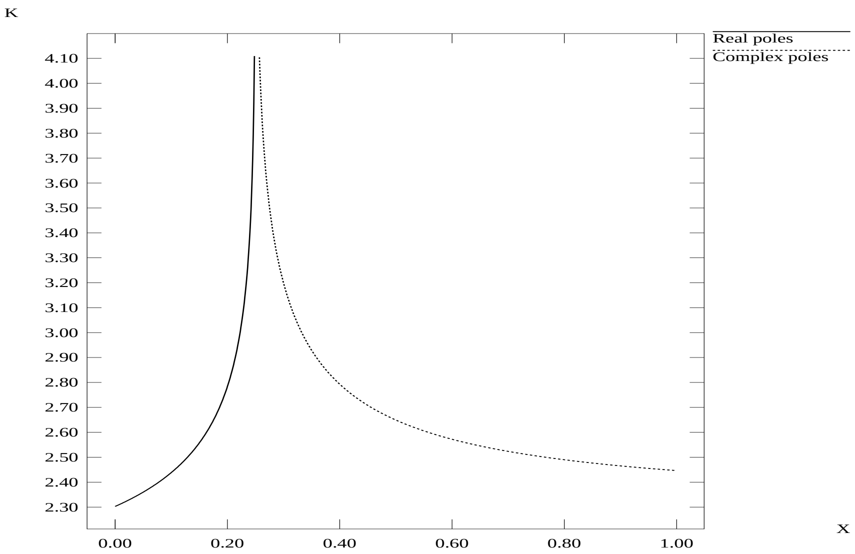

Figure 4: The variation of K with respect to the ratio $X = \frac{b_2}{b_1^2}$.

Complex Poles

Complex poles arise when $\sqrt{4M_2 - 3M_1^2} = \sqrt{b_1^2 - 4b_2} \leq 0$, i.e., $L > \frac{R^2C}{24}$. The time-domain response for complex poles is given by (see Appendix D):

$$v(t) = V_0 \left(1 - \frac{\sqrt{\alpha^2 + \beta^2}}{\beta} e^{-\alpha t} sin(\beta t + \rho)\right) \geq V_0 \left(1 - \frac{\sqrt{\alpha^2 + \beta^2}}{\beta} e^{-\alpha t}\right)$$

where $\rho = \tan^{-1}(\frac{\beta}{\alpha})$. Using the lower bound on the response, the delay at 90% threshold can again be estimated as a upper bound and is given by

$$\tau_{0.9} = \frac{1}{\alpha} \ln\left(10\sqrt{1+(\frac{\alpha}{\beta})^2}\right) = \frac{K_c}{\alpha}$$

where

$$K_c = \ln\left(10\sqrt{1+(\frac{\alpha}{\beta})^2}\right) = \ln\left(10\sqrt{\frac{4b_2/b_1^2}{4b_2/b_1^2-1}}\right)$$

An appropriate value of K_c (the subscript c indicates complex poles) can be chosen from the variation of K_c with respect to either the ratio $\frac{M_2}{M_1^2}$ or the ratio $\frac{b_2}{b_1^2}$. From the empirical study in Figure 4, we choose K_c heuristically as $K_c \approx \ln 10 = 2.3$ (the limiting value for K_c as $\frac{b_2}{b_1^2}$ tends to ∞); this gives

$$\tau_{0.9} \quad = \quad \frac{K_c}{\alpha} = 2K_c\frac{M_1^2 - M_2}{M_1} = 2K_c\frac{b_2}{b_1} = 0.38RC + 4.6\frac{L}{R} \tag{26}$$

In practice, k_r and K_c could also be found using a table lookup, based on the value of $\frac{b_2}{b_1^2}$.

Double Poles

Finally, a double pole occurs when is $b_1^2 = 4b_2$, i.e., $L = \frac{R^2C}{24}$. The double-pole response is

$$V(s) \quad = \quad \frac{V_0}{s}\frac{1}{1+b_1s+b_2s^2+\ldots} = \frac{V_0}{s}\frac{1}{b_2(s-s_1)^2} = V_0\left(\frac{1}{s} - \frac{1}{s-s_1} - \frac{2}{b_1}\frac{1}{(s-s_1)^2}\right)$$

where $s_1 = -\frac{b_1}{2b_2}$, and the time-domain response is given by $v(t) = V_0\left(1 - e^{ts_1} - \frac{2t}{b_1}e^{ts_1}\right)$. The delay at 90% threshold is obtained as

$$\tau_{0.9} = \frac{2b_2}{b_1}\ln\left(10(1+\frac{2T_{0.9}}{b_1})\right) = K_d\frac{2b_2}{b_1} = K_d\frac{b_1}{2}$$

(the subscript d indicates a double pole). This gives a recursive equation for K_d, i.e.,

$$K_d = \ln\left(10(1+\frac{2T_{0.9}}{b_1})\right) = \ln\left(10(1+K_d)\right)$$

from which $K_d \approx 3.9$. Thus, for the case of a double pole the 90% threshold delay is estimated as

$$\tau_{0.9} \approx 1.95b_1 = 0.975RC \tag{27}$$

which is independent of inductance value.

5.2 Distributed RLC Line with Capacitive Load

The New Delay model can be applied to arbitrary interconnection networks by extending it to the case of a distributed RLC line with general load. To compare with the previous development, here we consider capacitive loads only. Recall that the transfer function for a distributed RLC line with capacitive load is

$$\begin{aligned} H(s) &= \frac{1}{\cosh(\theta h) + \frac{Z_0}{Z_T}\cdot\sinh(\theta h)} \\ &= \frac{1}{\left(1+\frac{sC(R+sL)}{2}+\frac{s^2C^2(R+sL)^2}{4!}+\ldots\right)+sC_T\sqrt{\frac{R+sL}{sC}}\left(\sqrt{sC(R+sL)}+\frac{(sC(R+sL))^{\frac{3}{2}}}{3!}+\ldots\right)} \\ &= \frac{1}{1+s(\frac{RC}{2}+RC_T)+s^2(\frac{(RC)^2}{24}+\frac{R^2CC_T}{6}+\frac{LC}{2}+LC_T)+\ldots} \end{aligned} \tag{28}$$

Introducing the variable $C_{eq} = \frac{C}{2} + C_T$, we have

$$\begin{aligned} b_1 &= R(\frac{C}{2}+C_T) = RC_{eq} \\ b_2 &= \frac{R^2C}{6}(\frac{C}{4}+C_T)+L(\frac{C}{2}+C_T) = \frac{R^2C}{6}(C_{eq}-\frac{C}{4})+LC_{eq}. \end{aligned}$$

Real poles occur when $b_1^2 - 4b_2 \geq 0$, i.e.,

$$(RC_{eq})^2 - \frac{2}{3}R^2C(C_{eq}-\frac{C}{4}) - 4LC_{eq} \geq 0$$

which implies $L \leq \frac{R^2}{4}\left(C_{eq} - \frac{2}{3}C + \frac{C^2}{6C_{eq}}\right)$. A development identical to that in the preceding subsection yields

$$\tau_{0.9} = \frac{K_r}{|s_1|} = \frac{2K_rb_2}{b_1-\sqrt{b_1^2-4b_2}} = 2.61\frac{\frac{R^2C}{3}(C_{eq}-\frac{C}{4})+2LC_{eq}}{RC_{eq}-\sqrt{(RC_{eq})^2-\frac{2}{3}R^2C(C_{eq}-\frac{C}{4})-4LC_{eq}}} \tag{29}$$

Here, K_r was again estimated as $K_r \approx 2.61$ by reconciling with the known case of distributed RC line delay.

For complex poles, $b_1^2 - 4b_2 \leq 0$ and $L > \frac{R^2}{4}\left(C_{eq} - \frac{2}{3}C + \frac{C^2}{6C_{eq}}\right)$, implying 90% threshold delay

$$\tau_{0.9} = K_c\frac{1}{\alpha} = 2K_c\frac{b_2}{b_1} = 4.6\left(\frac{\frac{R^2C}{6}(C_{eq}-\frac{C}{4})+LC_{eq}}{RC_{eq}}\right) \tag{30}$$

For the double pole that occurs with $L = \frac{R^2}{4}\left(C_{eq} - \frac{2}{3}C + \frac{C^2}{6C_{eq}}\right)$, the 90% threshold delay is

$$\tau_{0.9} = \frac{2K_db_2}{b_1} = \frac{K_db_1}{2} = 1.95RC_{eq} \tag{31}$$

5.3 Interconnect Trees

An RLC network is called an *RLC tree* if it does not contain a closed path of resistors and inductors, i.e., all resistors and inductors are floating with respect to ground and all capacitors are connected to ground [Wya87]. We now consider RLC trees where each edge is modeled using distributed RLC 2-port parameters. Consider the voltage response and delay at an arbitrary output node k. The path from source S (root of the tree) to node k is denoted by $p(k)$ and is referred to as the *main path*. The edges/nodes not on the main path are the *off-path* edges/nodes. Elmore delay at node k for RC interconnect trees [RPH83, BKR93] is given by

$$T_{ED}(k) = R_S C_{T(S)} + \sum_{\forall i \in p(k)} R_i \left(\frac{C_i}{2} + C_{T(i)}\right)$$

where $T(S)$ is the tree rooted at source S, $T(i)$ is the subtree rooted at node i, R_S is the source resistance, and $C_{T(i)}$ is the total capacitance of the subtree rooted at node i. This model is widely used for interconnect tree analysis and synthesis, but since it cannot take into account the inductance of the interconnect line, the delay estimation may not be accurate and the trees synthesized could be sub-optimal. Using the New Delay model, we can obtain a new delay estimate for RLC interconnect trees, namely,

$$T_{0.9}(k) = 2.3 R_S C_{T(S)} + \sum_{\forall i \in p(k)} \tau_{rcd}(i)$$

where $\tau_{rcd}(i)$ indicates the 90% threshold delay for distributed RLC line with capacitive load for all the three cases depending on the value of the inductance of the tree edge i.

$$\begin{aligned}
\tau_{rcd}(i) &= 2.61 \frac{\frac{R^2C}{3}(C_{eq} - \frac{C}{4}) + 2LC_{eq}}{RC_{eq} - \sqrt{(RC_{eq})^2 - \frac{2}{3}R^2C(C_{eq} - \frac{C}{4}) - 4LC_{eq}}} && \text{for real poles} \\
\tau_{rcd}(i) &= 4.6 \left(\frac{\frac{R^2C}{6}(C_{eq} - \frac{C}{4}) + LC_{eq}}{RC_{eq}} \right) && \text{for complex poles} \\
\tau_{rcd}(i) &= 1.95 RC_{eq} && \text{for a double pole}
\end{aligned}$$

6 Conclusions

We have obtained response and delay estimates at any arbitrary location x of a distributed RLC line with general source and load impedances. We also obtain a closed form expression for the exact time-domain response at any location x of a distributed RC line. We thus obtain many useful advances (as well as concordances and corrections) with respect to previous delay estimates in the literature. We also present a New Delay model for distributed RLC lines which considers the effect of inductance, unlike previous delay estimates (Elmore delay); this extends to general interconnection trees.

We are currently extending these techniques into a unified methodology for response and delay estimation in distributed $RLCG$ interconnect structures. Also, these techniques can be used in developing non-uniform lumped segment models for distributed $RLCG$ interconnect lines with general source and load impedances. We are also working on extending these techniques for stable signal propagation [FC92] considering time-of-flight in the delay estimation. Finally, we have recently used the New Delay model for delay-driven routing tree construction, zero-skew routing, and delay estimation in nets spanning over multilayers (via modeling) [KM94b].

Appendix A: Solution of voltage response at any location x of a distributed line

The $ABCD$ parameters of a distributed line (Figure 1) are

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta h) & Z_0 \sinh(\theta h) \\ \frac{1}{Z_0} \sinh(\theta h) & \cosh(\theta h) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix}$$

where $V_1(s), I_1(s)$ and $V_2(s), I_2(s)$ are voltages and currents at the input and output ends of the line respectively. Equation (1) implies that voltage and current at any x can also be expressed as

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta x) & Z_0 \sinh(\theta x) \\ \frac{1}{Z_0} \sinh(\theta x) & \cosh(\theta x) \end{pmatrix} \begin{pmatrix} V_x(s) \\ I_x(s) \end{pmatrix} \tag{32}$$

where $V_x(s)$ and $I_x(s)$ are voltage and current at any location x of the line. From Equation (1) and Equation (32), we obtain

$$V_1(s) = V_2(s)\cosh(\theta h) + I_2(s) Z_0 \sinh(\theta h) = V_x(s)\cosh(\theta x) + I_x(s) Z_0 \sinh(\theta x) \tag{33}$$

$$I_1(s) = V_2(s)\frac{1}{Z_0}\sinh(\theta h) + I_2(s)\cosh(\theta h) = V_x(s)\frac{1}{Z_0}\sinh(\theta x) + I_x(s)\cosh(\theta x) \tag{34}$$

Multiplying Equation (33) by $\cosh(\theta x)$ and Equation (34) by $Z_0 \sinh(\theta x)$, then subtracting, we obtain

$$V_x(s) = V_2(s)\cosh(\theta(h-x)) + I_2(s) Z_0 \sinh(\theta(h-x))$$

One similarly obtains

$$I_x(s) = V_2(s)\frac{1}{Z_0}\sinh(\theta(h-x)) + I_2(s)\cosh(\theta(h-x))$$

and we express these two equations in matrix form as follows:

$$\begin{pmatrix} V_x(s) \\ I_x(s) \end{pmatrix} = \begin{pmatrix} \cosh(\theta(h-x)) & Z_0 \sinh(\theta(h-x)) \\ \frac{1}{Z_0} \sinh(\theta(h-x)) & \cosh(\theta(h-x)) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix} \tag{35}$$

Recall that similar expressions for $ABCD$ parameters at any x for a distributed line are given without derivation in [GK68, Sak83].

Appendix B: Exact time-domain response at any x for open-ended distributed lines

This Appendix calculates the time-domain response at any x for a distributed RC line. Using the response in the Laplace domain, Equation (5), and expanding cosh terms yields

$$\begin{aligned} V_x(s) &= \frac{V_0}{s} \cdot \frac{\cosh(\theta(h-x))}{\cosh(\theta h)} \\ &= \frac{V_0}{s} \cdot \frac{e^{\theta(h-x)} + e^{-\theta(h-x)}}{e^{\theta h} + e^{-\theta h}} = \frac{V_0}{s} \cdot \frac{e^{-\theta x} + e^{-\theta(2h-x)}}{1 + e^{-2\theta h}} \end{aligned} \tag{36}$$

The identity

$$\frac{1}{1+e^{-2\theta h}} = \sum_{n=0}^{\infty} \left(-e^{-2\theta h}\right)^n = \sum_{n=0}^{\infty} (-1)^n e^{-2n\theta h}$$

implies

$$\begin{aligned} V_x(s) &= \frac{V_0}{s} \cdot \left(e^{-\theta x} \sum_{n=0}^{\infty} (-1)^n e^{-2n\theta h} + e^{-\theta(2h-x)} \sum_{n=0}^{\infty} (-1)^n e^{-2n\theta h} \right) \\ &= \frac{V_0}{s} \cdot \left([e^{-\theta x} - e^{-\theta(2h+x)} + e^{-\theta(4h+x)} + \ldots] + [e^{-\theta(2h-x)} - e^{-\theta(4h-x)} + e^{-\theta(6h-x)} + \ldots] \right) \end{aligned}$$

Distributed RC Line

Substituting $\theta = \sqrt{src}$ for the distributed RC line and applying inverse Laplace transforms,

$$L\left\{ \frac{e^{-a\sqrt{s}}}{s} \right\}^{-1} = 1 - erf\left(\frac{a}{2\sqrt{t}}\right) = erfc\left(\frac{a}{2\sqrt{t}}\right)$$

Therefore, the time-domain response is

$$\begin{aligned} v_x(t) &= V_0 \left\{ \left[erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) - erfc\left(\sqrt{\frac{R_{2h+x} C_{2h+x}}{4t}}\right) + erfc\left(\sqrt{\frac{R_{4h+x} C_{4h+x}}{4t}}\right) + \ldots \right] \right. \\ & \quad + \left. \left[erfc\left(\sqrt{\frac{R_{2h-x} C_{2h-x}}{4t}}\right) - erfc\left(\sqrt{\frac{R_{4h-x} C_{4h-x}}{4t}}\right) + \ldots \right] \right\} \\ &= V_0 \left\{ erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) + \sum_{n=1}^{\infty} (-1)^n erfc\left(\sqrt{\frac{R_{2nh+x} C_{2nh+x}}{4t}}\right) \right. \\ & \quad + \left. \sum_{n=1}^{\infty} (-1)^{n-1} erfc\left(\sqrt{\frac{R_{2nh-x} C_{2nh-x}}{4t}}\right) \right\} \end{aligned} \tag{37}$$

where $R_{2nh\pm x} = (2nh \pm x)r$ and $C_{2nh\pm x} = (2nh \pm x)c$.

The response can be very well approximated by considering only few terms of the sum. For $x = h$ the response for the distributed RC line reduces to that given in [Mat93] and is given by

$$v_h(t) = 2V_0 \sum_{n=0}^{\infty}(-1)^n erfc\left(\sqrt{\frac{R_{(2n+1)h}C_{(2n+1)h}}{4t}}\right) = 2V_0 \sum_{n=0}^{\infty}(-1)^n erfc\left((2n+1)\sqrt{\frac{R_hC_h}{4t}}\right) \tag{38}$$

Distributed (Lossless) LC Line

Substituting $\theta = s\sqrt{lc}$ for the distributed (lossless) LC line and applying inverse Laplace transforms,

$$L\left\{\frac{e^{-as}}{s}\right\}^{-1} = u(t-a)$$

Therefore, the time-domain response is

$$\begin{aligned} v_x(t) &= V_0\left\{\left[u\left(t-\sqrt{L_xC_x}\right) - u\left(t-\sqrt{L_{2h+x}C_{2h+x}}\right) + u\left(t-\sqrt{L_{4h+x}C_{4h+x}}\right) + \dots\right]\right. \\ &\quad + \left.\left[u\left(t-\sqrt{L_{2h-x}C_{2h-x}}\right) - u\left(t-\sqrt{L_{4h-x}C_{4h-x}}\right) + \dots\right]\right\} \\ &= V_0\left\{u\left(t-\sqrt{L_xC_x}\right) + \sum_{n=1}^{\infty}(-1)^n u\left(t-\sqrt{L_{2nh+x}C_{2nh+x}}\right) + \sum_{n=1}^{\infty}(-1)^{n-1} u\left(t-\sqrt{L_{2nh-x}C_{2nh-x}}\right)\right\} \end{aligned} \tag{39}$$

where $L_{2nh\pm x} = (2nh \pm x)l$ and $C_{2nh\pm x} = (2nh \pm x)c$.

Appendix C: Calculation of Moments from the Transfer Function Polynomial

In a linear system the transfer function $H(s) = \frac{V_{out}(s)}{V_{in}(s)}$ gives the relationship between the output response $V_{out}(s)$ and the input response $V_{in}(s)$. The system transfer function $H(s)$ is related to the impulse response $h(t)$ by the equation

$$H(s) = \int_0^\infty e^{-st} h(t) dt$$

Without loss of generality, the transfer function for any linear system can be expressed as a ratio of polynomials in s, that is to say,

$$H(s) = K \frac{1 + a_1 s + a_2 s^2 + a_3 s^3 + ...}{1 + b_1 s + b_2 s^2 + b_3 s^3 + ...} \quad (40)$$

where K is the DC (zero frequency) gain. The i^{th} moment of the linear system is

$$M_i = \frac{1}{i!} \int_0^\infty t^i h(t) dt = \frac{(-1)^i}{i!} H^{(i)}(0) \quad (41)$$

where $H^{(i)}(0)$ is the i^{th} derivative of $H(s)$ at $s = 0$. Assuming $V_{out}(0) = 0$, the Laplace transform of the derivative of the output voltage response for a unit step input is $v'_{out}(t) \Leftrightarrow sV_{out}(s) = s \cdot \frac{1}{s} H(s) = H(s) \Leftrightarrow h(t)$; hence, the transfer function can also be written as

$$H(s) = \int_0^\infty e^{-st} v'_{out}(t) dt$$

Expanding e^{-st} into a Maclaurin series,

$$H(s) = \int_0^\infty v'_{out}(t) dt - \frac{s}{1!} \int_0^\infty t v'_{out}(t) dt + \frac{s^2}{2!} \int_0^\infty t^2 v'_{out}(t) dt - \frac{s^3}{3!} \int_0^\infty t^3 v'_{out}(t) dt + \dots$$

By identifying the integral quantities as moments M_0, M_1, M_2, M_3 etc. from Equation (41),

$$H(s) = (M_0 - sM_1 + s^2 M_2 - s^3 M_3 + ...) \quad (42)$$

Therefore, the moments can also be defined as

$$M_i = \frac{1}{i!} \int_0^\infty t^i v'_{out}(t) dt \quad (43)$$

In general, the moments of any system can be calculated using the definitions in the Laplace transform domain and the time domain as in Equations (41) and (42), or by comparing with Equation (43). For calculating the moments, consider the system transfer function given in Equation (40):

$$H(s) = K \frac{1 + a_1 s + a_2 s^2 + a_3 s^3 + ...}{1 + b_1 s + b_2 s^2 + b_3 s^3 + ...}$$

Expanding the denominator polynomial as an infinite series, we have

$$\begin{aligned}
H(s) &= K(1 + a_1 s + a_2 s^2 + a_3 s^3 + ...)(1 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4 + ...)^{-1} \\
&= K(1 + a_1 s + a_2 s^2 + a_3 s^3 + ...) \left(1 - b_1 s + (b_1^2 - b_2)s^2 - (b_3 + b_1^2 - 2b_1 b_2)s^3 + ...\right) \\
&= K[1 + s(a_1 - b_1) + s^2(a_2 - a_1 b_1 + b_1^2 - b_2) \\
&\quad + s^3(a_3 - a_2 b_1 + a_1(b_1^2 - b_2) - b_3 - b_1^3 + 2b_1 b_2) + ...]
\end{aligned}$$

Then, the definition of moments in Equation (43) yields

$$\begin{aligned}
M_0 &= K \\
M_1 &= K(b_1 - a_1) \\
M_2 &= K(a_2 - a_1 b_1 + b_1^2 - b_2) \\
M_3 &= K\left(b_1^3 + b_3 - 2b_1 b_2 - a_3 + a_2 b_1 - a_1(b_1^2 - b_2)\right) \\
M_4 &= K\left(a_4 - a_3 b_1 + a_2(b_1^2 - b_2) + a_1(b_1^2 + b_3 - 2b_1 b_2) + b_1^4 + b_2^2 + 2b_1 b_3 - b_4 - 3b_1^2 b_2\right)
\end{aligned}$$

Appendix D: Calculation of Voltage Response Using Moments

The transfer function of the system considering two dominant poles is

$$H(s) = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2}$$

where s_1, s_2 are the poles and k_1, k_2 are the coefficients corresponding to the poles. Assuming a step input, $V_{in}(t) = V_0 u(t)$, $V_{in}(s) = \frac{V_0}{s}$, and the output voltage is given by

$$V_{out}(s) = V_0 \frac{H(s)}{s}$$

Applying partial fractions and taking the inverse Laplace transform:

$$v_{out}(t) = \left[-\left(\frac{k_1}{s_1} + \frac{k_2}{s_2} \right) + \frac{k_1}{s_1} e^{s_1 t} + \frac{k_2}{s_2} e^{s_2 t} \right]$$

Using the boundary conditions $v_{out}(t \to \infty) = V_0$ and $v'_{out}(t = 0) = 0$, one can solve for s_1, s_2, k_1, k_2, i.e.,

$$-\left(\frac{k_1}{s_1} + \frac{k_2}{s_2} \right) = 1 \;\; and \;\; k_1 + k_2 = 0$$

yielding the time-domain response

$$v_{out}(t) = \left(1 + \frac{k_1}{s_1} e^{s_1 t} + \frac{k_2}{s_2} e^{s_2 t} \right)$$

By applying the definition of moments from Equation (43) to the two-pole transfer function, we get

$$\begin{aligned} \left(\frac{k_1}{s_1^2} + \frac{k_2}{s_2^2}\right) &= M_1 \\ \left(\frac{k_1}{s_1^3} + \frac{k_2}{s_2^3}\right) &= M_2 \end{aligned}$$

and solving the above equations yields

$$\begin{aligned} s_{1,2} &= \frac{2}{-M_1 \pm \sqrt{4M_2 - 3M_1^2}} \\ k_1 &= -k_2 = -\frac{1}{\sqrt{4M_2 - 3M_1^2}} \end{aligned}$$

Note that the poles of the system should be always in the left half of the $s-$plane for stable systems. The voltage response can be calculated for both real and complex poles as follows:

Case 1: Real Poles.

From the above equations,

$$k_1 = -\frac{s_1 s_2}{s_2 - s_1}$$

and

$$k_2 = \frac{s_1 s_2}{s_2 - s_1}$$

The voltage response for real poles is

$$v_{out}(t) = V_0(1 - \frac{s_2}{s_2 - s_1}e^{s_1 t} - \frac{s_1}{s_1 - s_2}e^{s_2 t}) \tag{44}$$

Case 2: Complex Poles.

Since the poles are complex we can express them in the form $s_1 = -\alpha + \jmath\beta$ and $s_2 = -\alpha - \jmath\beta$.

The voltage response is

$$\begin{aligned} v_{out}(t) &= V_0\left[1 - \frac{s_2}{s_2 - s_1}e^{s_1 t} - \frac{s_1}{s_1 - s_2}e^{s_2 t}\right] \\ &= V_0\left[1 - \frac{e^{-\alpha t}}{2}\left((1 + \frac{\alpha}{\jmath\beta})e^{\jmath\beta t} + (1 - \frac{\alpha}{\jmath\beta})e^{-\jmath\beta t}\right)\right] \\ &= V_0\left[1 - e^{-\alpha t}(\cos(\beta t) + \frac{\alpha}{\beta}\sin(\beta t))\right] \\ &= V_0\left[1 - \frac{\sqrt{\alpha^2 + \beta^2}}{\beta}e^{-\alpha t}\sin(\beta t + \rho)\right] \end{aligned}$$

where $\rho = \tan^{-1}(\frac{\beta}{\alpha})$ and

$$\alpha = \frac{M_1}{2(M_1^2 - M_2)}$$

$$\beta = \frac{\sqrt{3M_1^2 - 4M_2}}{2(M_1^2 - M_2)}$$

Appendix E: Definitions of Elmore Delay

Elmore delay [Elm48] is defined to be the first moment of the system impulse response, i.e., the coefficient of s in the system transfer function $H(s)$. The output response in Laplace domain for an impulse input is $H(s)$, so the impulse response in time-domain is $h(t)$. Therefore, Elmore delay is $T_{ED} = \int_0^\infty th(t)dt$.

Another definition of Elmore delay [Elm48] for step input is, the first moment of the derivative of the time-domain response, i.e., $T_{ED} = \int_0^\infty t\frac{dv(t)}{dt}dt = \int_0^\infty tv'(t)dt$. Assuming $V(s=0)=0$, the Laplace transform of the derivative of the output voltage response for a unit step input is $v'_{out}(t) \Leftrightarrow sV(s) - V(s = 0) = s \cdot \frac{1}{s}H(s) = H(s) \Leftrightarrow h(t)$. Therefore, $T_{ED} = \int_0^\infty tv'(t)dt = \int_0^\infty th(t)dt$. Also using integration by parts we can show that $T_{ED} = \int_0^\infty tv'(t)dt = \int_0^\infty (1 - v(t))dt$.

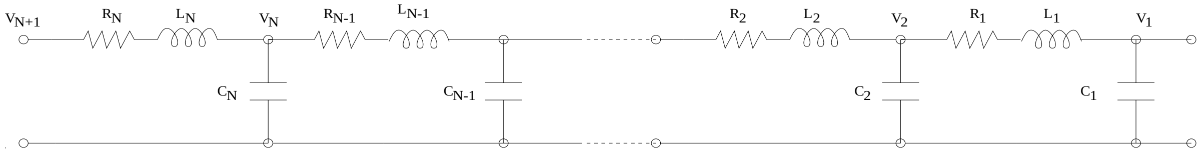

Figure 5: N-segment distributed RLC transmission line model.

Using the N-segmented distributed model (Figure 5) for the interconnect line and applying the Elmore definition of RC trees [RPH83] we get,

$$T_{ED} = b_1 = \sum_{j=1}^{N} C_j \sum_{i=j}^{N} R_i = \sum_{j=1}^{N} C_j R_{1j}$$

where R_{1j} is defined as the resistance of the unique path between the input node $(N+1)$ and output node (1), that is common with the unique path between the input node and node j.

For N uniform distributed **L** segments,

$$b_1 = \sum_{j=1}^{N} \frac{C}{N} \sum_{i=j}^{N} \frac{R}{N} = \frac{RC}{N^2} \sum_{j=1}^{N} (N - j + 1) = \frac{RC}{2} \frac{(N+1)}{N}$$

and thus

$$\lim_{N\to\infty} b_1 = \frac{RC}{2}.$$

Another approach for calculating the Elmore delay for a distributed RLC line is using the integration approach [RPH83]. Since the resistance and capacitance are distributed the Elmore delay for RC trees reduces to

$$T_{ED} = \sum_{j=1}^{N} C_j \sum_{i=j}^{N} R_i = \int_{x=0}^{x=h} r(x)cdx$$

where h is the length of the line and $r(x)$ is the cumulative resistance up to the length x, r, c are the resistance and capacitance per unit length and cdx is the infinitesimal value of capacitance at x. For uniformly distributed lines, $r(x) = rx$. Therefore,

$$T_{ED} = \int_{x=0}^{x=h} r(x)cdx = rc\frac{h^2}{2} = \frac{RC}{2}$$

The above Elmore definition is applicable for RC trees [RPH83] and RC meshes [Wya87] which have zero initial charge and whose response is always monotonic in time. Lin and Mead [LM84] extended the Elmore definition for RC meshes with nonmonotone step response, and Elmore delay for RC meshes with non-zero initial charge is discussed in [Hor84, CH87]. Note that Elmore delay is same for both RC trees/meshes and RLC trees/meshes because the inductance doesn't affect the coefficient of s in the transfer function.

Appendix F: Propagation Delay

The propagation delays of physical interconnects in the present VLSI circuits have a significant, dominant impact on the performance. Due to the increasing operating (chip) frequencies, the transmission line effects of the interconnections, for example, signal-delay, reflections, attenuation, non-linear loads and crosstalk, should be considered in modeling of interconnections. The total propagation delay of interconnections is due to the following two components of delay: (i) the electromagnetic propagation delay (T_{EM}) is due to the time taken for the electromagnetic wave to propagate from source to the destination, and (ii) interconnect rise time delay (T_R) is due to the distributed parameters of the interconnect line. The electromagnetic propagation delay is also referred as time of flight (T_{flight}).

The velocity of propagation of electromagnetic waves [Dwo79, LB90] in the interconnect line is given by

$$\nu = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\epsilon_r}}$$

where μ and ϵ are the permittivity and permeability in the medium, ϵ_r is the relative dielectric constant and c is the velocity of light in free space. At the same time, the velocity of electromagnetic waves in a *lossless* transmission line ($R = 0, G = 0$) [Dwo79] is given by

$$\nu = \frac{1}{\sqrt{lc}}$$

where l and c are the inductance and capacitance per unit length of the transmission line. Therefore, the electromagnetic propagation delay (T_{EM}) for an interconnect line of length h is given by

$$T_{EM} = \frac{h}{\nu}$$

For a given interconnect medium, the wave velocity ν is fixed, so the electromagnetic propagation delay increases linearly with length l, with this variation shown in Figure 6. Approximating the rise time delay (T_R) to the first moment [Elm48], we find it proportional to the product of resistance and capacitance of the line, i.e,

$$T_R = kRC = krch^2$$

where k is constant and R and C are the total resistance and capacitance of the interconnect line. So the rise time delay is proportional to length squared (Figure 6). From the definition of propagation delay, the voltage at the end of the transmission line will remain zero until T_{EM} and then starts rising.

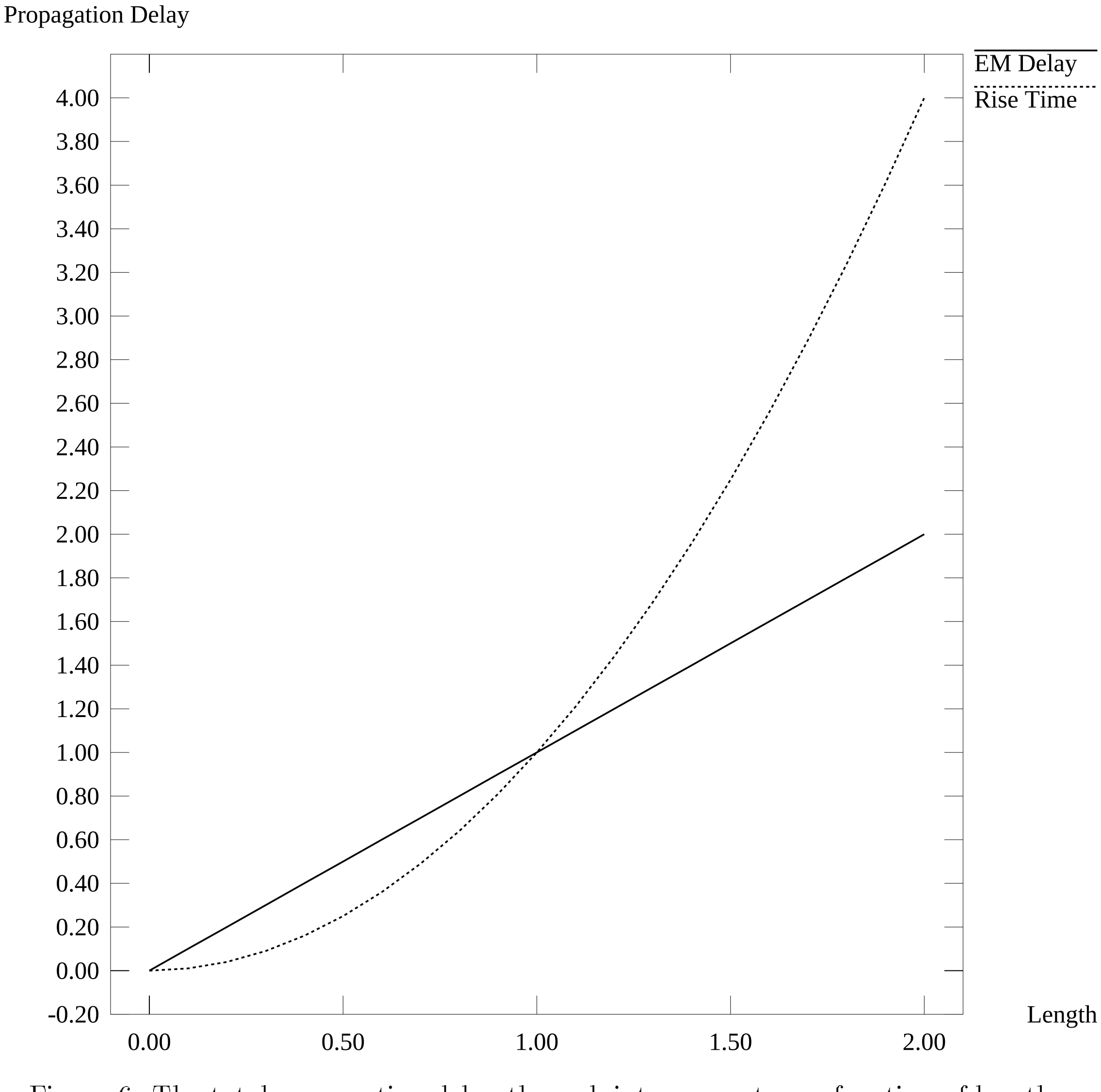

Figure 6: The total propagation delay through interconnect as a function of length.

Appendix G: Response for general series and parallel source impedance and general load impedance.

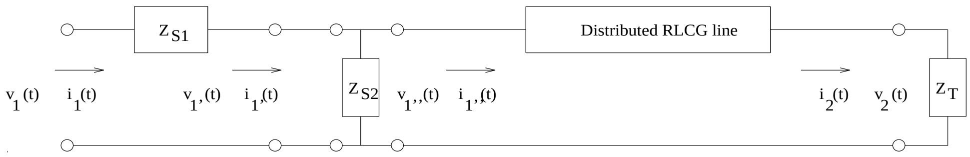

Figure 7: 2-port model of a distributed $RLCG$ line with series and parallel source impedances (Z_{S1}, Z_{S2}).

The $ABCD$ parameters for the series source impedance (Z_{S1}) (Figure 7) are

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} 1 & Z_{S1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_{1'}(s) \\ I_{1'}(s) \end{pmatrix} \tag{45}$$

Similarly the $ABCD$ parameters for the parallel source impedance (Z_{S2}) (Figure 7) are

$$\begin{pmatrix} V_{1'}(s) \\ I_{1'}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{Z_{S2}} & 1 \end{pmatrix} \begin{pmatrix} V_{1''}(s) \\ I_{1''}(s) \end{pmatrix} \tag{46}$$

Therefore, the total $ABCD$ parameters for the source are

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} 1 + \frac{Z_{S1}}{Z_{S2}} & Z_{S1} \\ \frac{1}{Z_{S2}} & 1 \end{pmatrix} \begin{pmatrix} V_{1''}(s) \\ I_{1''}(s) \end{pmatrix} \tag{47}$$

Substituting Equation (1) into the above Equation (47) and multiplying the matrices, we get

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} (1 + \frac{Z_{S1}}{Z_{S2}}) \cosh(\theta h) + \frac{Z_{S1}}{Z_0} \sinh(\theta h) & (1 + \frac{Z_{S1}}{Z_{S2}}) Z_0 \sinh(\theta h) + Z_{S1} \cosh(\theta h) \\ \frac{1}{Z_{S2}} \cosh(\theta h) + \frac{1}{Z_0} \sinh(\theta h) & \frac{Z_0}{Z_{S2}} \sinh(\theta h) + \cosh(\theta h) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix} \tag{48}$$

Therefore, the voltage at the beginning of the line is

$$V_1(s) = V_2(s) \left[(1 + \frac{Z_{S1}}{Z_{S2}}) \cosh(\theta h) + \frac{Z_{S1}}{Z_0} \sinh(\theta h) \right] + I_2(s) \left[(1 + \frac{Z_{S1}}{Z_{S2}}) Z_0 \sinh(\theta h) + Z_{S1} \cosh(\theta h) \right]$$

From the Equation (2), the voltage at any x is given by

$$V_x(s) = V_2(s) \cosh(\theta(h - x)) + I_2(s) Z_0 \sinh(\theta(h - x))$$

For general load $Z_T = R_T + sL_T + \frac{1}{sC_T}$, the transfer function at any x is obtained by substituting $V_2(s) = Z_T I_2(s)$ in the above equations and dividing them:

$$H(x, s) = \frac{V_x(s)}{V_1(s)} = \frac{\cosh(\theta(h - x)) + \frac{Z_0}{Z_T} \sinh(\theta(h - x))}{\left[(1 + \frac{Z_{S1}}{Z_{S2}}) \cosh(\theta h) + \frac{Z_{S1}}{Z_0} \sinh(\theta h) \right] + \frac{1}{Z_T} \left[(1 + \frac{Z_{S1}}{Z_{S2}}) Z_0 \sinh(\theta h) + Z_{S1} \cosh(\theta h) \right]} \tag{49}$$

Appendix H: Transfer function for distributed $RLCG$ line with resistive source and capacitive load.

The general expression for the transfer function at any x is given in Equation (17),

$$H(x,s) = \frac{V_x(s)}{V_0(s)} = \frac{\cosh(\theta(h-x)) + \frac{Z_0}{Z_T}\sinh(\theta(h-x))}{\left[\cosh(\theta h) + \frac{Z_S}{Z_0}\sinh(\theta h)\right] + \frac{1}{Z_T}\left[Z_0\sinh(\theta h) + Z_S\cosh(\theta h)\right]}$$

For a distributed $RLCG$ line the values of θ and Z_0 are given by

$$\theta(h-x) = \sqrt{(R+sL)(G+sC)(1-\frac{x}{h})^2}$$

$$Z_0 = \sqrt{\frac{R+sL}{G+sC}}$$

Substituting for θ, Z_0 and considering resistive source and capacitive load, the transfer function reduces to

$$H(x,s) = \left\{ \frac{\left(1 + \frac{(R+sL)(G+sC)(1-\frac{x}{h})^2}{2} + \frac{(R+sL)^2(G+sC)^2(1-\frac{x}{h})^4}{24} + \ldots\right) +}{(1+sR_SC_T)\left(1 + \frac{(R+sL)(G+sC)}{2} + \frac{(R+sL)^2(G+sC)^2}{24} + \ldots\right) +} \right.$$
$$\left. \frac{sC_T\sqrt{\frac{R+sL}{G+sC}}\left(\frac{\sqrt{(R+sL)(G+sC)(1-\frac{x}{h})^2}}{1} + \frac{(R+sL)(G+sC)(1-\frac{x}{h})^{2\frac{3}{2}}}{6} + \ldots\right)}{(R_s\sqrt{\frac{G+sC}{R+sL}} + sC_T\sqrt{\frac{R+sL}{G+sC}})\left(\frac{\sqrt{(R+sL)(G+sC)}}{1} + \frac{(R+sL)(G+sC)^{\frac{3}{2}}}{6} + \ldots\right)} \right\}$$

Expanding the cosh and sinh as infinite series we get

$$\begin{aligned}
\cosh(\theta h) &= \left(1 + \frac{(R+sL)(G+sC)}{2!} + \frac{(R+sL)^2(G+sC)^2}{4!} + \frac{(R+sL)^3(G+sC)^3}{6!} + \ldots\right) \\
&= \left[\left(1 + \frac{RG}{2} + \frac{R^2G^2}{24} + \ldots\right) + s\left(\frac{RC+LG}{2} + \frac{2RLG^2+2R^2CG}{24} + \frac{3R^2LG^3+3R^3CG^2}{720} + \ldots\right)\right. \\
&\quad \left. + s^2\left(\frac{LC}{2} + \frac{R^2C^2+L^2G^2+4RLCG}{24} + \frac{3RL^2G^3+3R^3C^2G+9R^2LCG^2}{720} + \ldots\right) + \ldots\right]
\end{aligned}$$

$$\begin{aligned}
\cosh(\theta(h-x)) &= \left[\left(1 + \frac{RG(1-\frac{x}{h})^2}{2} + \ldots\right) + s\left(\frac{(RC+LG)(1-\frac{x}{h})^2}{2} + \frac{(2RLG^2+2R^2CG)(1-\frac{x}{h})^4}{24}\right.\right. \\
&\quad \left. + \frac{(3R^2LG^3+3R^3CG^2)(1-\frac{x}{h})^6}{720} + \ldots\right) \\
&\quad + s^2\left(\frac{LC(1-\frac{x}{h})^2}{2} + \frac{(R^2C^2+L^2G^2+4RLCG)(1-\frac{x}{h})^4}{24}\right.
\end{aligned}$$

$$+\ \frac{(3RL^2G^3 + 3R^3C^2G + 9R^2LCG^2)(1-\frac{x}{h})^6}{720} + \ldots \Big) + \ldots \Big]$$

$$\begin{aligned}
\frac{Z_0}{Z_T}\sinh(\theta h) &= sC_T\sqrt{\frac{R+sL}{G+sC}}\left(\frac{\sqrt{(R+sL)(G+sC)}}{1} + \frac{(R+sL)(G+sC)^{\frac{3}{2}}}{3!} + \frac{(R+sL)(G+sC)^{\frac{5}{2}}}{5!} + \ldots\right) \\
&= \left(sC_T(R+sL) + \frac{sC_T(R+sL)^2(G+sC)}{6} + \frac{sC_T(R+sL)^3(G+sC)^2}{120} + \ldots\right) \\
&= \Bigg[s\left(RC_T + \frac{R^2C_TG}{6} + \frac{R^3C_TG^2}{120} + \ldots\right) \\
&\quad + s^2\left(LC_T + \frac{2RLC_TG + R^2CC_T}{6} + \frac{2R^3CC_TG + 3R^2LC_TG^2}{120} + \ldots\right) + \ldots\Bigg]
\end{aligned}$$

$$\begin{aligned}
\frac{Z_0}{Z_T}\sinh(\theta(h-x)) &= \Bigg[s\left(RC_T(1-\frac{x}{h}) + \frac{R^2C_TG(1-\frac{x}{h})^3}{6} + \frac{R^3C_TG^2(1-\frac{x}{h})^5}{120} + \ldots\right) + s^2\Big(LC_T(1-\frac{x}{h}) \\
&\quad + \frac{2RLC_TG + R^2CC_T(1-\frac{x}{h})^3}{6} + \frac{2R^3CC_TG + 3R^2LC_TG^2(1-\frac{x}{h})^5}{120} + \ldots\Big) + \ldots\Bigg]
\end{aligned}$$

$$\begin{aligned}
\frac{Z_S}{Z_0}\sinh(\theta h) &= R_s\sqrt{\frac{G+sC}{R+sL}}\left(\frac{\sqrt{(R+sL)(G+sC)}}{1} + \frac{(R+sL)(G+sC)^{\frac{3}{2}}}{3!} + \frac{(R+sL)(G+sC)^{\frac{5}{2}}}{5!} + \ldots\right) \\
&= \left(R_S(G+sC) + \frac{R_S(R+sL)(G+sC)^2}{6} + \frac{R_S(R+sL)^2(G+sC)^3}{120} + \ldots\right) \\
&= \Bigg[\left(R_SG + \frac{R_SRG^2}{6} + \frac{R_SR^2G^3}{120} + \ldots\right) + s\Big(R_SC + \frac{2R_SRCG + LG^2}{6} \\
&\quad + \frac{3R_SR^2CG^2 + 2R_SRLG^3}{120} + \ldots\Big) + s^2\Big(\frac{R_SRC^2 + 2R_SLCG}{6} \\
&\quad + \frac{R_SL^2G^3 + 3R_SRC^2G + 6R_SRLCG^2}{120} + \ldots\Big) + \ldots\Bigg]
\end{aligned}$$

The numerator and denominators polynomials of the transfer function can be obtained by substituting expansions for cosh and sinh. Unlike in the previous cases, the coefficients of s terms of the transfer function for the distributed $RLCG$ line are themselves infinite series. The coefficients of the numerator polynomial of the transfer function are:

$$a_0 = \left(1 + \frac{RG(1-\frac{x}{h})^2}{2} + \frac{R^2G^2(1-\frac{x}{h})^4}{24} + \ldots\right)$$

$$
\begin{aligned}
a_1 &= \Bigg\{\Bigg(\frac{(RC+LG)(1-\frac{x}{h})^2}{2} + \frac{(2RLG^2+2R^2CG)(1-\frac{x}{h})^4}{24} + \frac{(3R^2LG^3+3R^3CG^2)(1-\frac{x}{h})^6}{720} + \ldots\Bigg) \\
&+ \Bigg(RC_T(1-\frac{x}{h}) + \frac{R^2C_TG(1-\frac{x}{h})^3}{6} + \frac{R^3C_TG^2(1-\frac{x}{h})^5}{120} + \ldots\Bigg)\Bigg\} \\
&\approx \Bigg\{\Bigg(\frac{(RC+LG)(1-\frac{x}{h})^2}{2} + \frac{(2RLG^2+2R^2CG)(1-\frac{x}{h})^4}{24}\Bigg) + \Bigg(RC_T(1-\frac{x}{h}) + \frac{R^2C_TG(1-\frac{x}{h})^3}{6}\Bigg)\Bigg\} \\
a_2 &= \Bigg\{\Bigg(\frac{LC(1-\frac{x}{h})^2}{2} + \frac{(R^2C^2+L^2G^2+4RLCG)(1-\frac{x}{h})^4}{24} + \frac{(3RL^2G^3+3R^3C^2G+9R^2LCG^2)(1-\frac{x}{h})^6}{720} + \ldots\Bigg) \\
&+ \Bigg(LC_T(1-\frac{x}{h}) + \frac{2RLC_TG+R^2CC_T(1-\frac{x}{h})^3}{6} + \frac{2R^3CC_TG+3R^2LC_TG^2(1-\frac{x}{h})^5}{120} + \ldots\Bigg)\Bigg\} \\
&\approx \Bigg\{\Bigg(\frac{LC(1-\frac{x}{h})^2}{2} + \frac{(R^2C^2+L^2G^2+4RLCG)(1-\frac{x}{h})^4}{24}\Bigg) + \Bigg(LC_T(1-\frac{x}{h}) + \frac{2RLC_TG+R^2CC_T(1-\frac{x}{h})^3}{6}\Bigg)\Bigg\}
\end{aligned}
$$

Similarly, the coefficients of the denominator polynomial of the transfer function are:

$$
\begin{aligned}
b_0 &= \Bigg\{(1+\frac{RG}{2}+\frac{R^2G^2}{24}+\frac{R^3G^3}{720}+\ldots) + (R_SG + \frac{R_SRG^2}{6} + \frac{R_SR^2G^3}{120}+\ldots)\Bigg\} \\
b_1 &= \Bigg\{R_SC_T\Bigg(1+\frac{RG}{2}+\frac{R^2G^2}{24}+\ldots\Bigg) + \Bigg(\frac{RC+LG}{2} + \frac{2RLG^2+2R^2CG}{24} + \frac{3R^2LG^3+3R^3CG^2}{720}+\ldots\Bigg) \\
&+ \Bigg(RC_T + \frac{R^2C_TG}{6} + \frac{R^3C_TG^2}{120}+\ldots\Bigg) + \Bigg(R_SC + \frac{2R_SRCG+LG^2}{6} + \frac{3R_SR^2CG^2+2R_SRLG^3}{120}+\ldots\Bigg)\Bigg\} \\
&\approx \Bigg\{R_SC_T\Bigg(1+\frac{RG}{2}+\frac{R^2G^2}{24}\Bigg) + \Bigg(\frac{RC+LG}{2} + \frac{2RLG^2+2R^2CG}{24}\Bigg) + \Bigg(RC_T + \frac{R^2C_TG}{6}\Bigg) \\
&+ \Bigg(R_SC + \frac{2R_SRCG+LG^2}{6}\Bigg)\Bigg\} \\
b_2 &= \Bigg\{\Bigg(\frac{LC}{2} + \frac{R^2C^2+L^2G^2+4RLCG}{24} + \frac{3RL^2G^3+3R^3C^2G+9R^2LCG^2}{720}+\ldots\Bigg) \\
&+ R_SC_T\Bigg(\frac{RC+LG}{2} + \frac{2RLG^2+2R^2CG}{24} + \frac{3R^2LG^3+3R^3CG^2}{720}+\ldots\Bigg) \\
&+ \Bigg(LC_T + \frac{2RLC_TG+R^2CC_T}{6} + \frac{2R^3CC_TG+3R^2LC_TG^2}{120}+\ldots\Bigg) \\
&+ \Bigg(\frac{R_SRC^2+2R_SLCG}{6} + \frac{R_SL^2G^3+3R_SRC^2G+6R_SRLCG^2}{120}+\ldots\Bigg)\Bigg\} \\
&\approx \Bigg\{\Bigg(\frac{LC}{2} + \frac{R^2C^2+L^2G^2+4RLCG}{24}\Bigg) + R_SC_T\Bigg(\frac{RC+LG}{2} + \frac{2RLG^2+2R^2CG}{24}\Bigg) \\
&+ \Bigg(LC_T + \frac{2RLC_TG+R^2CC_T}{6}\Bigg) + \Bigg(\frac{R_SRC^2+2R_SLCG}{6}\Bigg)\Bigg\}
\end{aligned}
$$

References

[BKR93] K. D. Boese, A. B. Kahng and G. Robins, "High-Performance Routing Trees With Identified Critical Sinks", *Proc. 30th ACM/IEEE Design Automation Conf.*, June 1993, pp. 182-187.

[CH87] C-Y. Chu and M. L. Horowitz, "Charge-Sharing Models for Switch-level Simulation", *IEEE Trans. on CAD*, Nov. 1987, pp. 1053-1061.

[Dwo79] L. N. Dworsky, *Modern Transmission Line Theory and Applications*, Wiley, 1979.

[Elm48] W. C. Elmore, "The Transient Response of Damped Linear Networks with Particular Regard to Wideband Amplifiers", *Journal of Applied Physics* 19, Jan. 1948, pp. 55-63.

[FC92] R. C. Frye and H. Z. Chen, "Optimal Self-Damped Lossy Transmission Line Interconnections for Multichip Modules", *IEEE Trans. on Circuits and Systems-II*, Nov. 1992, pp. 765-771.

[GK68] M. S. Ghausi and J. J. Kelly, *Introduction to Distributed-Parameter Networks: With Application to Integrated Circuits*, New York: Holt, Rinehart and Winston, 1968.

[Hor84] M. A. Horowitz, "Timing Models for MOS Circuits", *PhD Thesis*, Stanford University, Jan. 1984.

[KM93] A. B. Kahng and S. Muddu, "Delay Estimation of Trees using Two-pole method and Optimal Equivalent Circuits", *UCLA CS Dept. TR-930035*, October 1993.

[KM94b] A. B. Kahng and S. Muddu, *manuscript in preparation*, April 1994.

[LB90] S. I. Long and S. E. Butner, *Gallium Arsenide Digital Integrated Circuit Design*, McGraw-Hill, 1990.

[LM84] T. Lin and C. A. Mead "Signal Delay in General RC Networks", *IEEE Trans. on CAD*, Oct. 1984, pp. 331-349.

[Mat93] H. L. Mattes, "Behavior of a Single Transmission Line Stimulated With a Step Function", *manuscript*, August 1993.

[RPH83] J. Rubinstein, P. Penfield and M. A. Horowitz, "Signal Delay in RC Tree Networks", *IEEE Trans. on CAD* 2(3), July 1983, pp. 202-211.

[Sak83] T. Sakurai, "Approximation of Wiring Delay in MOSFET LSI", *IEEE Journal of Solid-State Circuits*, Aug. 1983, Vol.18, No.4, pp. 418-426.

[Sak93] T. Sakurai, "Closed-Form Expressions for Interconnection Delay, Coupling, and Crosstalk in VLSI's", *IEEE Trans. on Electron Devices*, Jan. 1993, pp. 118-124.

[WE92] N. H. E. Weste and K. Eshraghian, *Principles of CMOS VLSI Design: A Systems Perspective*, 2nd ed., Addison-Wesley, 1992.

[Wil71] A. Wilnai, "Open-Ended RC Line Model Predicts MOSFET IC Response", *EDN*, Dec. 15, 1971, pp. 53-54.

[Wya87] J. L. Wyatt, Jr, " Signal Propagation Delay in RC models for Interconnect", *Circuit Analysis, Simulation and Design*, Elsevier Science Publishers, 1987.

[ZSTGC92] D. Zhou, S. Su, F. Tsui, D. S. Gao and J. S. Cong, "A Simplified Synthesis of Transmission Lines with A Tree Structure", *Intl. Journal of Analog Integrated Circuits and Signal Processing*, Jan. 1994, pp. 19-30.