Appendix: Laplace Transform Identity

We present the following identity which is useful in calculating time-domain expressions for the distributed RC line from the corresponding transform domain expressions.[4]

Theorem: Let $f(t)$ and $F(s)$ be a Laplace transform pair. Then

$$\frac{F(\sqrt{s})}{\sqrt{s}} \Longleftrightarrow \frac{1}{\sqrt{\pi t}} \int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} f(u) du$$

Proof: Any function $f(u)$ can be expressed using the delta function as

$$f(u) = \int_{u_0=0}^{\infty} f(u)\delta(u-u_0)du_0 = \int_{u_0=0}^{\infty} f(u_0)\delta(u-u_0)du_0$$

Applying this to the time-domain function which is to be inverted, we get

$$\begin{aligned} \frac{1}{\sqrt{\pi t}} \int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} f(u) du &= \frac{1}{\sqrt{\pi t}} \int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} du \left(\int_{u_0=0}^{\infty} f(u_0)\delta(u-u_0)du_0 \right) \\ &= \frac{1}{\sqrt{\pi t}} \int_{u_0=0}^{\infty} f(u_0)du_0 \int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} \delta(u-u_0)du \\ &= \frac{1}{\sqrt{\pi t}} \int_{u_0=0}^{\infty} f(u_0) e^{\frac{-u_0^2}{4t}} du_0 \end{aligned} \tag{23}$$

Taking the Laplace transform for the above equation,

$$\begin{aligned} \int_{t=0}^{\infty} e^{-st} dt \left(\frac{1}{\sqrt{\pi t}} \int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} f(u) du \right) &= \int_{t=0}^{\infty} e^{-st} dt \frac{1}{\sqrt{\pi t}} \int_{u_0=0}^{\infty} f(u_0) e^{\frac{-u_0^2}{4t}} du_0 \\ &= \int_{u_0=0}^{\infty} f(u_0)du_0 \int_{t=0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{\frac{-u_0^2}{4t}} e^{-st} dt \end{aligned} \tag{24}$$

Note [4] that the integral with respect to t is a Laplace transform of

$$\int_{t=0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{\frac{-u_0^2}{4t}} e^{-st} dt = \frac{e^{-u_0\sqrt{s}}}{\sqrt{s}}$$

Substituting in the above equation and using the definition of Laplace transform, we get

$$\begin{aligned} \int_{t=0}^{\infty} e^{-st} dt \left(\frac{1}{\sqrt{\pi t}} \int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} f(u) du \right) &= \int_{u_0=0}^{\infty} f(u_0) \frac{e^{-u_0\sqrt{s}}}{\sqrt{s}} du_0 \\ &= \frac{1}{\sqrt{s}} \int_{u_0=0}^{\infty} e^{-u_0\sqrt{s}} f(u_0) du_0 \\ &= \frac{F(\sqrt{s})}{\sqrt{s}} \end{aligned} \tag{25}$$

Therefore, the transform pair is given by

$$\frac{1}{\sqrt{\pi t}} \int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} f(u) du \Longleftrightarrow \frac{F(\sqrt{s})}{\sqrt{s}}$$

[4] We offer the following proof only because we have found an incorrect statement of this identity in the Schaum's Outline reference, "Mathematical Handbook of Formulas and Tables", by Murray R. Spiegel.

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The total response obtained above is exactly the same as the one given in Equation (7) and the reflected diffusion components in Section 5 also match the corresponding terms in the above infinite series. The time-domain response for this case is described in Section 5.

From the above discussion it is clear that both the diffusion equation approach and the 2-port transform domain approach can be made to yield the same result. Even though the 2-port methodology has been widely known, the diffusion equation approach affords the insight to express the total voltage as a sum of reflected diffusion components. We believe that the diffusion equation approach can be easily and effectively be extended to interconnect trees by making use of coefficients of reflection and transmission.

7 Conclusions

In conclusion, we have proposed new methods based on addition of diffusion components for calculating the time-domain response for a finite-length distributed RC line. The result obtained for the open-ended line is identical to that derived in the classical transform domain by [8, 14], and corrects an erroneous analysis in [6, 7]. We have extended the technique for more realistic cases with source resistance and capacitive load. The approximate analytical expression derived for the finite RC line with capacitive load, using only four reflection components, is different from the previous method of [14]. We have also presented new approximate analytical expressions for the voltage response in a finite RC line with resistive source and capacitive load. The delay estimates using this method (approximating with only three diffusion components) are very close to SPICE-computed delays. Our method is simple, and can achieve higher accuracy in the response by computing additional terms using numerical techniques. The analysis of RC lines using the new technique can be extended to general interconnection trees by considering both reflection and transmission components.

Finite RC line with resistive source and capacitive load

The total voltage on the line for the case of resistive source $Z_S = R_S$ and capacitive load $Z_L = \frac{1}{sC_L}$ can be reduced from Equation (17) to the form

$$V_{Tot}(x,s) = \frac{V_0}{s} \cdot \frac{\cosh(\theta(h-x)) + sC_L Z_0 \sinh(\theta(h-x))}{\left[(1+R_S sC_L)\cosh(\theta h) + (\frac{R_S}{Z_0} + sC_L Z_)\sinh(\theta h)\right]}$$

Substituting $\frac{Z_0}{Z_L} = q\sqrt{s}$, $\frac{R_S}{Z_0} = R_S\sqrt{s\frac{C_h}{R_h}} = p\sqrt{s}$ and $R_S C_L = pq$, we get

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s} \cdot \frac{(1+q\sqrt{s})e^{\theta(h-x)} + (1-q\sqrt{s})e^{-\theta(h-x)}}{(1+spq+q\sqrt{s}+p\sqrt{s})e^{\theta h} + (1+spq-q\sqrt{s}-p\sqrt{s})e^{-\theta h}} \\ &= \frac{V_0}{s} \cdot \frac{(1+q\sqrt{s})e^{\theta(h-x)} + (1-q\sqrt{s})e^{-\theta(h-x)}}{(1+p\sqrt{s})(1+q\sqrt{s})e^{\theta h} + (1-p\sqrt{s})(1-q\sqrt{s})e^{-\theta h}} \end{aligned}$$

Identifying the coefficients of reflection at the source and load as

$$\Gamma_S = -\frac{1-p\sqrt{s}}{1-p\sqrt{s}} \qquad \Gamma_L = \frac{1-q\sqrt{s}}{1-q\sqrt{s}}$$

and expressing the response expression in terms of the coefficients of reflections, we get

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s} \cdot \frac{1}{1+p\sqrt{s}} \cdot \frac{e^{-\theta x} + \Gamma_L e^{-\theta(2h-x)}}{1-\Gamma_S\Gamma_L e^{-2\theta h}} \\ &= \frac{V_0}{s(1+p\sqrt{s})} \cdot \left[e^{-\theta x} + \Gamma_L e^{-\theta(2h-x)}\right] \left[\sum_{n=0}^{\infty} {\Gamma_S}^n {\Gamma_L}^n e^{-2n\theta h}\right] \end{aligned}$$

Substituting $\theta = \sqrt{src}$ and expanding the terms, we have

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s(1+p\sqrt{s})} \left[\sum_{n=0}^{\infty} {\Gamma_S}^n {\Gamma_L}^n e^{-(2nh+x)\sqrt{src}} + \sum_{n=0}^{\infty} {\Gamma_S}^n {\Gamma_L}^{n+1} e^{-(2(n+1)h-x)\sqrt{src}}\right] \\ &= \frac{V_0}{s(1+p\sqrt{s})} \left[e^{-x\sqrt{src}} + \Gamma_L e^{-(2h-x)\sqrt{src}} + \Gamma_S\Gamma_L e^{-(2h+x)\sqrt{src}} + \Gamma_S\Gamma_L^2 e^{-(4h-x)\sqrt{src}} + \ldots\right] \end{aligned} \tag{22}$$

Let the first term be defined as

$$V_I(x,s) = \frac{V_0}{s(1+p\sqrt{s})} e^{-x\sqrt{src}}$$

so that the total voltage can be written as

$$V_{Tot}(x,s) = V_I(x,s) + \sum_{i=1}^{\infty} V_{R_i}(x,s)$$

where the i^{th} reflection gives

$$V_{R_i}(x,s) = \begin{cases} \Gamma_L^{\frac{i}{2}}(s)\Gamma_S^{\frac{i}{2}}(s)V_I(ih+x,s) & \text{for } i \text{ even} \\ \Gamma_L^{\frac{i+1}{2}}(s)\Gamma_S^{\frac{i-1}{2}}V_I(h(i+1)-x,s) & \text{for } i \text{ odd} \end{cases}$$

The response can be very well approximated by considering only few terms of the sum. For $x = h$ the response for the distributed RC line reduces to that given in [11] and is given by

$$v_{Tot}(h,t) = 2V_0 \sum_{n=0}^{\infty} (-1)^n erfc\left(\sqrt{\frac{R_{(2n+1)h} C_{(2n+1)h}}{4t}}\right) = 2V_0 \sum_{n=0}^{\infty} (-1)^n erfc\left((2n+1)\sqrt{\frac{R_h C_h}{4t}}\right) \tag{20}$$

Finite RC line with capacitive load

The voltage response for the line with capacitive load $Z_L = \frac{1}{sC_L}$ reduces to

$$V_{Tot}(x,s) = \frac{V_0}{s} \cdot \frac{\cosh(\theta(h-x)) + \frac{Z_0}{Z_L}\sinh(\theta(h-x))}{\cosh(\theta h) + \frac{Z_0}{Z_L}\sinh(\theta h)}$$

Substituting $\frac{Z_0}{Z_L} = C_L\sqrt{s\frac{R_h}{C_h}} = q\sqrt{s}$ and expressing the hyperbolic functions in terms of exponential functions, we get

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s} \cdot \frac{[e^{\theta(h-x)} + e^{-\theta(h-x)}] + q\sqrt{s}[e^{\theta(h-x)} - e^{-\theta(h-x)}]}{[e^{\theta h} + e^{-\theta h}] + q\sqrt{s}[e^{\theta h} + e^{-\theta h}]} \\ &= \frac{V_0}{s} \cdot \frac{(1+q\sqrt{s})e^{\theta(h-x)} + (1-q\sqrt{s})e^{\theta(h-x)}}{(1+q\sqrt{s})e^{\theta h} + (1-q\sqrt{s})e^{\theta h}} \end{aligned}$$

Identifying the coefficient of reflection at the load as

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1-q\sqrt{s}}{1-q\sqrt{s}}$$

and substituting in the total response equation above, we obtain

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s} \cdot \frac{e^{\theta(h-x)} + \Gamma_L e^{-\theta(h-x)}}{e^{\theta h} + \Gamma_L e^{-\theta h} e^{\theta h}} \\ &= \frac{V_0}{s} \cdot \frac{e^{-\theta x} + \Gamma_L e^{-\theta(2h-x)}}{1 + \Gamma_L e^{-2\theta h}} \\ &= \frac{V_0}{s} \cdot [e^{-\theta x} + \Gamma_L e^{-\theta(2h-x)}][\sum_{n=0}^{\infty}(-\Gamma_L e^{-2\theta h})^n] \end{aligned}$$

Substituting $\theta = \sqrt{src}$ and expanding the terms, we have

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s}\left[\sum_{n=0}^{\infty}(-1)^n {\Gamma_L}^n e^{-(2nh+x)\sqrt{src}} + \sum_{n=0}^{\infty}(-1)^n {\Gamma_L}^{n+1} e^{-(2(n+1)h-x)\sqrt{src}}\right] \\ &= \frac{V_0}{s}\left[e^{-x\sqrt{src}} + \Gamma_L e^{-(2h-x)\sqrt{src}} - e^{-(2h+x)\sqrt{src}} - e^{-(4h-x)\sqrt{src}} + \ldots\right] \end{aligned} \tag{21}$$

We can observe that the terms of this infinite series are exactly the same as those used in Section 4 for various diffusion reflection components. The time-domain solution can be calculated as in Section 4.

where $Z_0 = \sqrt{\frac{R+sL}{sC}}$, $\theta h = \sqrt{sRC}$ and $V_{Tot}(x,s)$ represents the total voltage at any position x on the line. For a step input at source, $V_S(s) = \frac{V_0}{s}$. We now obtain the response for the cases of open-ended line, capacitive load line, and resistive source and capacitive load line.

Open-ended finite RC line

For the case of open-ended line with ideal source the voltage response for a step input is

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s} \cdot \frac{\cosh(\theta(h-x))}{\cosh(\theta h)} \\ &= \frac{V_0}{s} \cdot \frac{e^{\theta(h-x)} + e^{-\theta(h-x)}}{e^{\theta h} + e^{-\theta h}} = \frac{V_0}{s} \cdot \frac{e^{-\theta x} + e^{-\theta(2h-x)}}{1 + e^{-2\theta h}} \end{aligned} \tag{18}$$

The identity

$$\frac{1}{1+e^{-2\theta h}} = \sum_{n=0}^{\infty} \left(-e^{-2\theta h}\right)^n = \sum_{n=0}^{\infty} (-1)^n e^{-2n\theta h}$$

implies

$$\begin{aligned} V_{Tot}(x,s) &= \frac{V_0}{s} \cdot \left(e^{-\theta x} \sum_{n=0}^{\infty} (-1)^n e^{-2n\theta h} + e^{-\theta(2h-x)} \sum_{n=0}^{\infty} (-1)^n e^{-2n\theta h} \right) \\ &= \frac{V_0}{s} \cdot \left([e^{-\theta x} - e^{-\theta(2h+x)} + e^{-\theta(4h+x)} + \ldots] + [e^{-\theta(2h-x)} - e^{-\theta(4h-x)} + e^{-\theta(6h-x)} + \ldots] \right) \end{aligned}$$

Substituting $\theta = \sqrt{src}$ for the distributed RC line and applying inverse Laplace transforms,

$$L\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}^{-1} = 1 - erf\left(\frac{a}{2\sqrt{t}}\right) = erfc\left(\frac{a}{2\sqrt{t}}\right)$$

Therefore, the time-domain response is

$$\begin{aligned} v_{Tot}(x,t) &= V_0 \left\{ \left[erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) - erfc\left(\sqrt{\frac{R_{2h+x} C_{2h+x}}{4t}}\right) + erfc\left(\sqrt{\frac{R_{4h+x} C_{4h+x}}{4t}}\right) + \ldots \right] \right. \\ &\quad \left. + \left[erfc\left(\sqrt{\frac{R_{2h-x} C_{2h-x}}{4t}}\right) - erfc\left(\sqrt{\frac{R_{4h-x} C_{4h-x}}{4t}}\right) + \ldots \right] \right\} \\ &= V_0 \left\{ erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) + \sum_{n=1}^{\infty} (-1)^n erfc\left(\sqrt{\frac{R_{2nh+x} C_{2nh+x}}{4t}}\right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} (-1)^{n-1} erfc\left(\sqrt{\frac{R_{2nh-x} C_{2nh-x}}{4t}}\right) \right\} \end{aligned} \tag{19}$$

where $R_{2nh\pm x} = (2nh \pm x)r$ and $C_{2nh\pm x} = (2nh \pm x)c$.

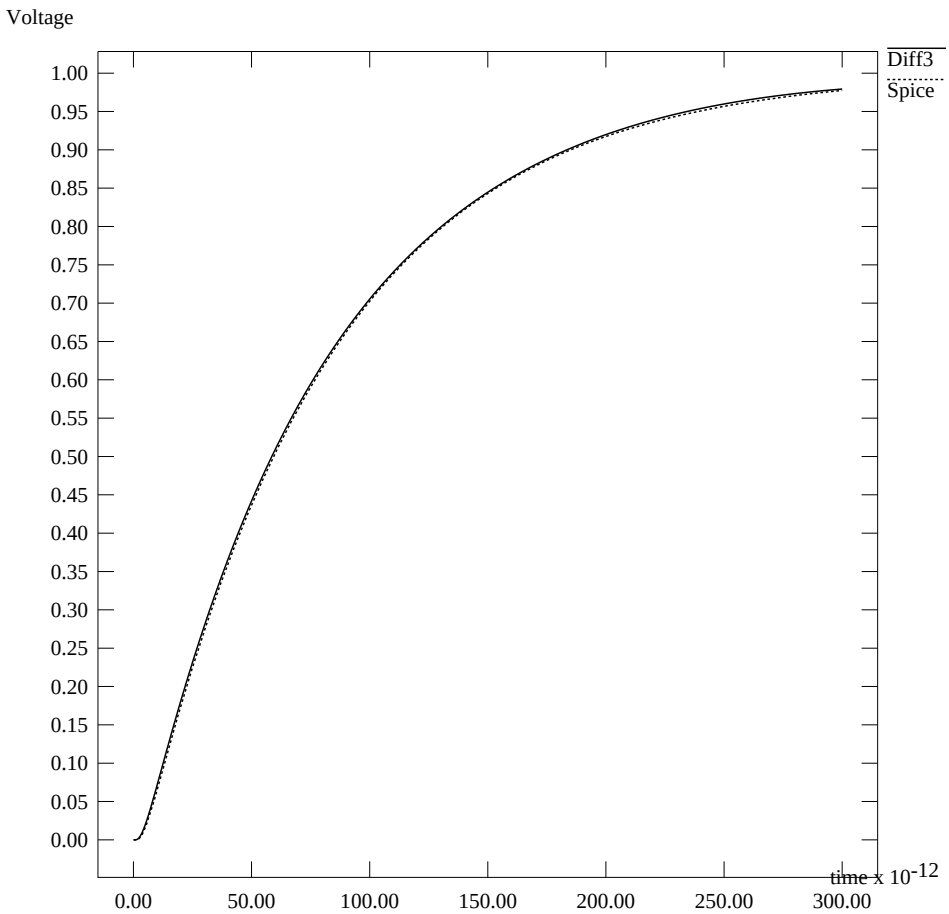

Figure 9: Voltage response at the end of finite RC line with source resistance $R_S = 2R_h$ and load capacitance $C_L = C_h$ using both **Diff3** model and SPICE. The time constant of the line is $R_hC_h = 15.0$ psec.

6 Transform Domain Analysis for Distributed RC Lines

In this section we try to derive the total voltage on the distributed RC line using a different approach, i.e., using the 2-port transfer function of the line. This analysis can be used to conclude that the sum of all of the diffusion components discussed above exactly corresponds to the total voltage response on the line. However, note that previous expansions of the response are different, and do not give the same physical intuition of reflections that we achieve here.

The transfer function for the interconnect model shown in Figure 3 is

$$H(x,s) = \frac{V_{Tot}(x,s)}{V_S(s)} = \frac{\cosh(\theta(h-x)) + \frac{Z_0}{Z_L}\sinh(\theta(h-x))}{\left[\cosh(\theta h) + \frac{Z_S}{Z_0}\sinh(\theta h)\right] + \frac{1}{Z_L}\left[Z_0\sinh(\theta h) + Z_S\cosh(\theta h)\right]} \tag{17}$$

$$= \frac{-V_0(1-q\sqrt{s})^2}{s(1+q\sqrt{s})^3} e^{\frac{-(2h+x)}{h}\sqrt{sR_hC_h}}$$

Let

$$\begin{aligned} F_9(s) &= \frac{-V_0(1-qs)^2}{s(1+qs)^3} e^{-bs} \\ &= V_0 \left[-\frac{1}{s} + \frac{1}{(s+1/q)} + \frac{4/q^2}{(s+1/q)^3} \right] e^{-bs} \end{aligned}$$

where $b = \frac{-(2h+x)}{h}\sqrt{R_hC_h}$, and the corresponding time-domain function is

$$f_9(t) = V_0 \left[-1 + e^{-\frac{(t-b)}{q}} + \frac{2}{q^2}(t-b)^2 e^{-\frac{(t-b)}{q}} \right]$$

Substituting for the time-domain function in the identity in the Appendix, the voltage response $v_{R_2}(x,t)$ is given by

$$v_{R_2}(x,t) = \frac{-V_0}{\sqrt{\pi t}} \int_{x=b}^{\infty} e^{\frac{-x^2}{4t}} \left[-1 + e^{-\frac{(t-b)}{q}} + \frac{2}{q^2}(t-b)^2 e^{-\frac{(t-b)}{q}} \right] dx$$

The above response can be obtained by using numerical integration methods. The total voltage on the line for this special case, considering two terms, is

$$\begin{aligned} v_{Tot}(x,t) &\approx v_I(x,t) + v_{R_1}(x,t) \\ &= V_0 erfc\left(\frac{x}{h}\sqrt{\frac{R_hC_h}{4t}} \right) + V_0 erfc\left(\frac{2h-x}{h}\sqrt{\frac{R_hC_h}{4t}} \right) \\ &\quad - V_0 e^{L1(C_L,1,1)} erfc(L2(C_L,1,1)) - \frac{4V_0}{q}\sqrt{\frac{t}{\pi}} e^{(L1(C_L,1,1)-L2(C_L,1)^2)} \\ &\quad + 2V_0 L3(C_L,1,1) e^{L1(C_L,1,1)} erfc(L2(C_L,1,1)) \end{aligned} \tag{16}$$

We have used the above **Diff3** approximation and computed the output response at the end of the line ($x = h$) for a distributed line with various values of source resistance and load capacitance. As shown in Table 2, the results are comparable to those obtained from SPICE simulation. The results correspond to the case of $C_L = C_h$, for which the ratio $\frac{p}{q} = \frac{R_S}{R_h}\frac{C_h}{C_L}$ reduces to $\frac{p}{q} = \frac{R_S}{R_h}$. Note that the results for $p = 0$ in Table 2 correspond to the entry $\frac{C_L}{C_h} = 1.0$ in Table 1, i.e., the case of ideal source and load capacitance equal to total line capacitance. Also note that for larger values of load impedance, we may need to consider more terms (or components) of the diffusion response. In general, the diffusion components need not be evaluated analytically, but instead can be computed by numerical integration methods. Figure 9 gives a comparison of voltage response between SPICE and **Diff3** model for the case of source resistance $R_S = 2R_h$ and load capacitance $C_L = C_h$.

Ratio ($\frac{p}{q} = \frac{R_S}{R_h}$)	10% Threshold delays(R_hC_h)		50% Threshold delays(R_hC_h)	
	SPICE	**Diff3**	**SPICE**	**Diff3**
0.0	0.289	0.287	1.088	1.085
0.25	0.390	0.385	1.448	1.455
0.5	0.469	0.465	1.808	1.810
1.0	0.601	0.600	2.530	2.535
2.0	0.859	0.840	3.907	3.910
Ratio ($\frac{p}{q} = \frac{R_S}{R_h}$)	**63.2% Threshold delays(R_hC_h)**		**90% Threshold delays(R_hC_h)**	
	SPICE	**Diff3**	**SPICE**	**Diff3**
0.0	1.500	1.510	3.260	3.250
0.25	2.010	2.010	4.320	4.330
0.5	2.511	2.515	5.430	5.440
1.0	3.530	3.550	7.712	7.000
2.0	5.510	5.510	12.290	12.240

Table 2: A comparison of delay values between SPICE and the analytical expression computed from diffusion analysis. We considered the case of $C_L = C_h$, for which the ratio $\frac{p}{q} = \frac{R_S}{R_h}\frac{C_h}{C_L}$ reduces to $\frac{p}{q} = \frac{R_S}{R_h}$.

to the form

$$\begin{aligned} V_{R_1}(x,s) &= \Gamma_L V_I(2h-x,s) \\ &= \frac{-V_0(1-q\sqrt{s})}{s(1+q\sqrt{s})^2} e^{\frac{-(2h-x)}{h}\sqrt{sR_hC_h}} \end{aligned}$$

To obtain the time-domain response, we consider the function

$$\begin{aligned} F_8(s) &= \frac{-V_0(1-qs)}{s(1+qs)^2} e^{-bs} \\ &= V_0\left[\frac{1}{s} - frac1(s+1/q) - \frac{2/q}{(s+1/q)^2}\right] e^{-bs} \end{aligned}$$

where $b = \frac{-(2h-x)}{h}\sqrt{R_hC_h}$. The response can be obtained using the above time-domain function in the identity given in the Appendix, i.e.,

$$\begin{aligned} v_{R_1}(x,t) &= V_0 erfc\left(\frac{2h-x}{h}\sqrt{\frac{R_hC_h}{4t}}\right) \\ &\quad - V_0 e^{L1(C_L,1,1)} erfc(L2(C_L,1,1)) - \frac{4V_0}{q}\sqrt{\frac{t}{\pi}} e^{(L1(C_L,1,1)-L2(C_L,1)^2)} \\ &\quad + 2V_0 L3(C_L,1,1) e^{L1(C_L,1,1)} erfc(L2(C_L,1,1)) \end{aligned}$$

We can similarly obtain the second diffusion component as

$$V_{R_2}(x,s) = \Gamma_L\Gamma_S V_I(2h+x,s)$$

The third reflected diffusion component is given by

$$\begin{aligned} V_{R_3}(x,s) &= \Gamma_L^2 \Gamma_S V_I(4h-x,s) \\ &= \frac{-V_0(1-q\sqrt{s})^2(1-p\sqrt{s})}{s(1+q\sqrt{s})^2(1+p\sqrt{s})^2} e^{\frac{-(4h-x)}{h}\sqrt{sR_hC_h}} \end{aligned}$$

To obtain the time-domain response, we consider the function

$$\begin{aligned} F_7(s) &= \frac{-V_0(1-qs)^2(1-ps)}{s(1+qs)^2(1+ps)^2} e^{-bs} \\ &= V_0 \left[-\frac{1}{s} + \frac{4qp(3q+p)}{(p-q)^3(s+1/q)} + \frac{4(p+q)}{(p-q)^2(s+1/q)^2} + \frac{2(p+q)^2}{p(p-q)^2(s+1/p)^2} \right. \\ &\quad \left. - \frac{(q^2+8qp-p^2)(p+q)}{(p-q)^3(s+1/p)} \right] e^{-bs} \end{aligned}$$

where $b = \frac{-(4h-x)}{h}\sqrt{R_hC_h}$. Therefore, the time-domain expression of the voltage response can be obtained as

$$\begin{aligned} v_{R_3}(x,t) &= -V_0 erfc\left(\frac{4h-x}{h}\sqrt{\frac{R_hC_h}{4t}}\right) \\ &\quad + \frac{4V_0pq(3q+p)}{(p-q)^3} e^{L1(C_L,2,1)} erfc(L2(C_L,2,1)) \\ &\quad - \frac{V_0(p+q)(q^2-p^2+8qp)}{(p-q)^3} e^{L1(R_S,2,1)} erfc(L2(R_S,2,1)) \\ &\quad + \frac{4(p+q)}{(p-q)^2}\sqrt{\frac{t}{\pi}} e^{(L1(C_L,2,1)-L2(C_L,2,1)^2)} - \frac{2(p+q)^2}{p(p-q)^2} L3(C_L,2,1) e^{L1(C_L,2,1)} erfc(L2(C_L,2,1)) \\ &\quad + \frac{2V_0(p+q)^2}{p(p-q)^2}\sqrt{\frac{t}{\pi}} e^{(L1(R_S,2,1)-L2(R_S,2,1)^2)} - \frac{2(p+q)^2}{p(p-q)^2} L3(R_S,2,1) e^{L1(R_S,2,1)} erfc(L2(R_S,2,1)) \end{aligned}$$

If we approximate the total voltage by considering only up to these three reflected diffusion components, we obtain our **Diff3** model for the RC line with resistive source and capacitive load:

$$v_{Tot}(x,t) \approx v_I(x,t) + v_{R_1}(x,t) + v_{R_2}(x,t) + v_{R_3}(x,t)$$

Of course, the above expressions are valid only when $p \neq q$ because the voltage response has terms with $(p-q)$ in the denominator.

Special Case: We now consider the special case of $p = q$, where each diffusion component of the response must be reevaluated. The voltage response of the first diffusion component reduces

where $b = \frac{-(2h-x)}{h}\sqrt{R_h C_h}$. Expanding the function using partial fractions and taking the inverse Laplace transform, we get

$$f_5(t) = V_0 \left[1 + (\frac{2q}{p-q})e^{-(t-b)/q} - (\frac{p+q}{p-q})e^{-(t-b)/p}\right] U(t-b)$$

The time-domain expression for this first reflected diffusion component is

$$\begin{aligned}
v_{R_1}(x,t) &= V_0 erfc\left(\frac{2h-x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\
&\quad + \frac{2V_0 q}{p-q} e^{\left((\frac{C_h}{C_L})^2 \frac{t}{R_h C_h} + \frac{(2h-x)C_h}{hC_L}\right)} erfc\left(\frac{C_h}{C_L}\sqrt{\frac{t}{R_h C_h}} + \frac{(2h-x)}{2h}\sqrt{\frac{R_h C_h}{t}}\right) \\
&\quad - V_0(\frac{p+q}{p-q}) e^{\left((\frac{R_h}{R_S})^2 \frac{t}{R_h C_h} + \frac{(2h-x)R_h}{hR_S}\right)} erfc\left(\frac{R_h}{R_S}\sqrt{\frac{t}{R_h C_h}} + \frac{(2h-x)}{2h}\sqrt{\frac{R_h C_h}{t}}\right) \\
&= V_0 erfc\left(\frac{2h-x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\
&\quad + \frac{2V_0 q}{p-q} e^{L1(C_L,1,1)} erfc(L2(C_L,1,1) - V_0(\frac{p+q}{p-q}) e^{L1(R_S,1,1)} erfc(L2(R_S,1,1)
\end{aligned}$$

We can similarly calculate the time-domain expressions for the other reflected diffusion components. The second reflected diffusion component is given by

$$\begin{aligned}
V_{R_2}(x,s) &= \Gamma_L \Gamma_S V_I(2h+x,s) \\
&= \frac{-V_0(1-q\sqrt{s})(1-p\sqrt{s})}{s(1+q\sqrt{s})(1+p\sqrt{s})^2} e^{\frac{-(2h+x)}{h}\sqrt{sR_h C_h}}
\end{aligned}$$

Let

$$\begin{aligned}
F_6(s) &= \frac{-V_0(1-qs)(1-ps)}{s(1+qs)(1+ps)^2} e^{-bs} \\
&= V_0\left[-\frac{1}{s} + \frac{2q(p+q)}{(p-q)^2(s+1/q)} + \frac{2(p+q)}{p(p-q)(s+1/p)^2} - \frac{(q^2+4qp-p^2)}{(p-q)^2(s+1/p)}\right] e^{-bs}
\end{aligned}$$

where $b = \frac{-(2h+x)}{h}\sqrt{R_h C_h}$. Substituting the time-domain expression of the function in the identity given in Appendix, the voltage response of the second reflect component is

$$\begin{aligned}
v_{R_2}(x,t) &= -V_0 erfc\left(\frac{2h+x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\
&\quad + \frac{2V_0 q(p+q)}{(p-q)^2} e^{L1(C_L,1,0)} erfc(L2(C_L,1,0)) - \frac{V_0(q^2-p^2+4pq)}{(p-q)^2} e^{L1(R_S,1,0)} erfc(L2(R_S,1,0)) \\
&\quad + \frac{4(p+q)}{p(p-q)}\sqrt{\frac{t}{\pi}} e^{(L1(R_S,1,0) - L2(R_S,1,0)^2)} - \frac{2(p+q)}{p(p-q)} L3(R_S,1,0) e^{L1(R_S,1,0)} erfc(L2(R_S,1,0))
\end{aligned}$$

where $F_3(s) = \frac{V_0}{s(1+ps)}$. The time-domain expression for the voltage at node A can be computed by again applying the identity in the Appendix:

$$\begin{aligned} v_A(t) &= \frac{1}{\sqrt{\pi t}} \int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} f_3(x)dx = \frac{1}{\sqrt{\pi t}} \int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} (1 - e^{-\frac{x}{p}})dx \\ &= V_0 \left[1 - e^{\frac{t}{p^2}} erfc\left(\frac{\sqrt{t}}{p}\right)\right] \end{aligned} \quad (15)$$

Then, the voltage response for the first incident diffusion can be computed using Equation (4) and the input voltage at the beginning of the line, i.e.,

$$V_I(x,s) = V_A(s)e^{-\frac{x}{h}\sqrt{R_h C_h}} = \frac{V_0}{s(1+p\sqrt{s})} e^{-\frac{x}{h}\sqrt{sR_h C_h}} = \frac{F_4(\sqrt{s})}{\sqrt{s}}.$$

where

$$\begin{aligned} F_4(s)7 = \; & \frac{V_0}{s(1+ps)} e^{-fracxh\sqrt{R_h C_h}} \\ & = V_0 \left[\frac{1}{s} - \frac{1}{(s+1/p)}\right] e^{-fracxh\sqrt{R_h C_h}} \end{aligned}$$

and the corresponding time-domain expression is

$$f_4(t) = \left[1 - e^{-frac(t-b)p}\right] U(t-b)$$

The voltage response due to the first incident diffusion component is obtained as

$$\begin{aligned} v_I(x,t) &= V_0 erfc\left(\frac{x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ &\quad - V_0 e^{\left((\frac{R_h}{R_S})^2 \frac{t}{R_h C_h} + \frac{xR_h}{hR_S}\right)} erfc\left(\frac{R_h}{R_S}\sqrt{\frac{t}{R_h C_h}} + \frac{x}{2h}\sqrt{\frac{R_h C_h}{t}}\right) \\ &= V_0 erfc\left(\frac{x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) - V_0 e^{L1(R_S,0,0)} erfc(L2(R_S,0,0)) \end{aligned}$$

From this incident diffusion response and the reflection coefficient at the load, the voltage response of the first reflected diffusion component can be obtained as

$$\begin{aligned} V_{R_1}(x,s) &= \Gamma_L V_I(2h-x,s) \\ &= \frac{V_0(1-q\sqrt{s})}{s(1+q\sqrt{s})(1+p\sqrt{s})} e^{\frac{-(2h-x)}{h}\sqrt{sR_h C_h}} \end{aligned}$$

To obtain the time-domain response, we consider the function

$$\begin{aligned} F_5(s) &= \frac{V_0(1-qs)}{s(1+qs)(1+ps)} e^{-bs} \\ &= V_0 \left[\frac{1}{s} + \frac{2q}{(p-q)(s+1/q)} - \frac{(p+q)}{(p-q)(s+1/p)}\right] e^{-bs} \end{aligned}$$

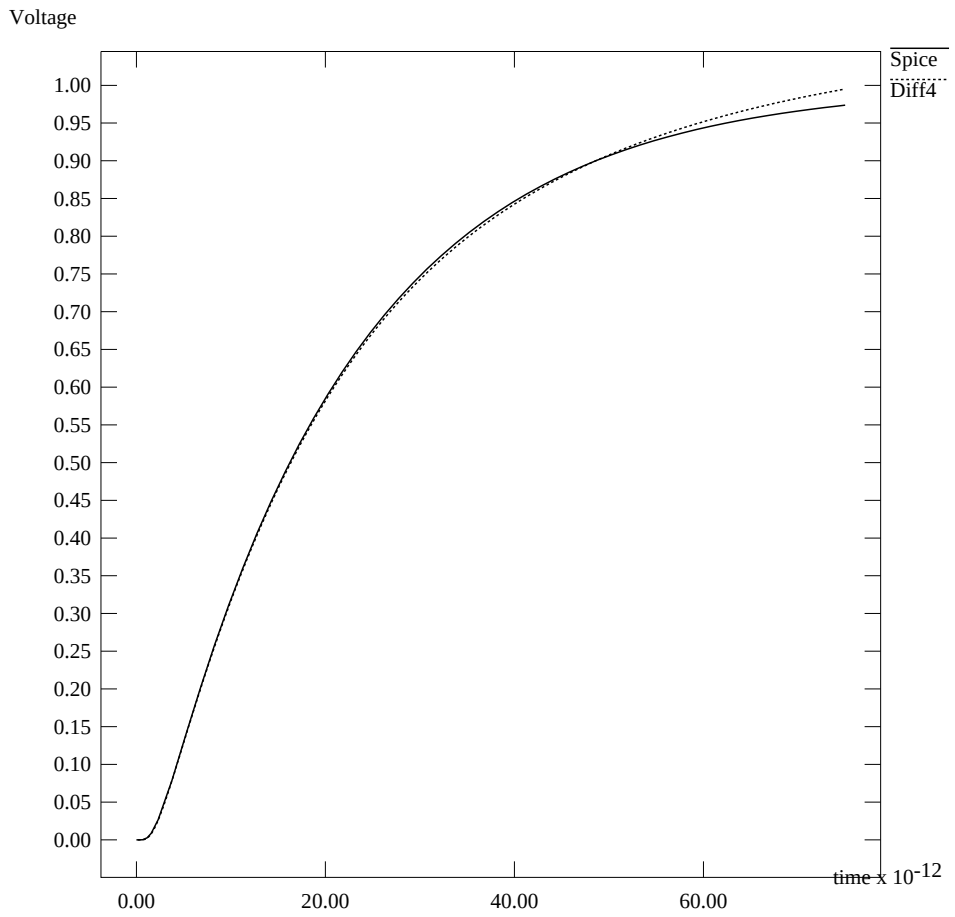

Figure 7: Voltage response at the end of finite RC line with load factor $\frac{C_L}{C_h} = 1.0$ using **Diff4** model and SPICE. The time constant of the line is $R_h C_h = 15.0$ psec.

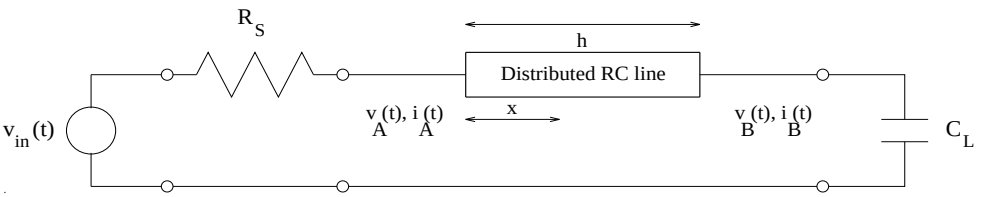

Figure 8: Distributed RC line of length h with a resistive source R_S and a capacitive load C_L.

where $p = R_S\sqrt{\frac{C_h}{R_h}} = \frac{R_S}{R_h}\sqrt{R_h C_h}$. The reflection coefficient at the load is again that of Equation (12). The voltage response at the beginning of the line (node A in the Figure 8) is $V_A(s) = V_{in}(s)\frac{Z_0}{Z_0+R_S}$. Considering a step input of magnitude V_0 and substituting for $Z_0 = \sqrt{\frac{R_h}{sC_h}}$, we get

$$V_A(s) = \frac{V_0}{s}\frac{Z_0}{Z_0 + R_S} = \frac{V_0}{s(1 + p\sqrt{s})} = \frac{F_3(\sqrt{s})}{\sqrt{s}}$$

Since some terms of the above integral cannot be expressed in analytical form, we will leave the response in the integral form and use numerical integration techniques for evaluation. Such a procedure can be applied to calculate the voltage response due to other reflected components, so that their contribution can be included in the total response. For simplicity, we approximate the total response by considering only up to the first four reflections, which yields

$$\begin{aligned} v_{Tot}(x,t) \approx\ & v_{open}(x,t) \\ & - 2V_0 e^{L1(C_L,1,1)} erfc(L2(C_L,1,1)) + 2V_0 e^{L1(C_L,1,0)} erfc(L2(C_L,1,0)) \\ & + \frac{8V_0 C_h}{\sqrt{\pi} C_L}\sqrt{\frac{t}{R_h C_h}} e^{(L1(C_L,2,1)-L2(C_L,2,1)^2)} - 4V_0 L3(C_L,2,1) e^{L1(C_L,2,1)} erfc(L2(C_L,2,1)) \\ & - \frac{8V_0 C_h}{\sqrt{\pi} C_L}\sqrt{\frac{t}{R_h C_h}} e^{(L1(C_L,2,0)-L2(C_L,2,0)^2)} + 4V_0 L3(C_L,2,0) e^{L1(C_L,2,0)} erfc(L2(C_L,2,0)) \end{aligned} \tag{13}$$

where the open-ended line response is

$$\begin{aligned} v_{open}(x,t) =\ & V_0 erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) + V_0 erfc\left(\frac{(2h-x)}{h}\sqrt{\frac{R_h C_h}{4t}}\right) - V_0 erfc\left(\frac{(2h+x)}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ & - V_0 erfc\left(\frac{4h-x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) + V_0 erfc\left(\frac{4h+x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \end{aligned} \tag{14}$$

We call this approximation in Equation (13) the **Diff4** model; Table 1 compares delay estimates at different threshold values for a wide range of capacitive loads, versus SPICE and the method of [14]. Observe that delay estimates using new diffusion equation approach are very close to the SPICE-computed delays (and to the method of [14] which is based on the first 10 poles of the response), even though only four reflections were considered. Figure 7 gives a comparison of voltage response between SPICE and the **Diff4** model for the case of load factor $\frac{C_L}{C_h} = 1.0$

5 Finite RC Line with Resistive Source and Capacitive Load

Finally, we consider a finite-length distributed RC line with both a resistive source and a capacitive load C_L, as shown in Figure 8. The reflection coefficient at the source for an impedance of $Z_S = R_S$ is

$$\Gamma_S(s) = \frac{Z_S - Z_0}{Z_S + Z_0} = -\frac{(1 - p\sqrt{s})}{(1 + p\sqrt{s})}$$

Substituting for $b = (\frac{4h-x}{h})\sqrt{R_h C_h}$ and $q = \frac{C_L}{C_h}\sqrt{R_h} C_h$, we get

$$\begin{aligned} v_{R_3}(x,t) &= -V_0 erfc\left(\frac{4h-x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ &\quad + \frac{8V_0 C_h}{\sqrt{\pi} C_L}\sqrt{\frac{t}{R_h C_h}} e^{(L1(C_L,2,1)-L2(C_L,2,1)^2)} \\ &\quad - 4V_0 L3(C_L,2,1) e^{L1(C_L,2,1)} erfc(L2(C_L,2,1)) \end{aligned}$$

Similarly, the voltage response of the fourth reflection at the source is

$$V_{R_4}(x,s) = {\Gamma_S}^2 {\Gamma_L}^2 V_I(4h+x,s) = {\Gamma_L}^2 V_I(4h+x,s)$$

and the corresponding time-domain response is

$$\begin{aligned} v_{R_4}(x,t) &= V_0 erfc\left(\frac{4h+x}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\ &\quad - \frac{8V_0 C_h}{\sqrt{\pi} C_L}\sqrt{\frac{t}{R_h C_h}} e^{(L1(C_L,2,0)-L2(C_L,2,0)^2)} \\ &\quad + 4V_0 L3(C_L,2,0) e^{L1(C_L,2,0)} erfc(L2(C_L,2,0)) \end{aligned}$$

Extending, we may calculate the voltage response of the fifth reflected diffusion component as

$$\begin{aligned} V_{R_5}(x,s) &= {\Gamma_S}^2 {\Gamma_L}^3 V_I(6h-x,s) \\ &= \frac{V_0}{s}\left(\frac{1-q\sqrt{s}}{1+q\sqrt{s}}\right)^3 e^{-frac(6h-x)h\sqrt{sR_h C_h}} = \frac{F_3(\sqrt{s})}{\sqrt{s}} \end{aligned}$$

where

$$\begin{aligned} F_3(s) &= \frac{V_0}{s}\left(\frac{1-qs}{1+qs}\right)^3 e^{-bs} \\ &= V_0\left[\frac{1}{s} - \frac{8/q^2}{(s+1/q)^3} + \frac{4/q}{(s+1/q)^2} - \frac{2}{(s+1/q)}\right] e^{-bs} \end{aligned}$$

with $b = \frac{(6h-x)}{h}\sqrt{R_h C_h}$. The corresponding time-domain function can be obtained as

$$f_3(t) = V_0\left[1 - 2e^{-\frac{(t-b)}{q}} + \frac{4(t-b)}{q}e^{-\frac{(t-b)}{q}} - \frac{4(t-b)^2}{q^2}e^{-\frac{(t-b)}{q}}\right] U(t-b)$$

Substituting for the time-domain function in the identity in the Appendix, the voltage response $v_{R_5}(x,t)$ is given by

$$v_{R_5}(x,t) = \frac{-V_0}{\sqrt{\pi t}}\int_{x=b}^{\infty} e^{\frac{-x^2}{4t}}\left[1 - 2e^{-\frac{(t-b)}{q}} + \frac{4(t-b)}{q}e^{-\frac{(t-b)}{q}} - \frac{4(t-b)^2}{q^2}e^{-\frac{(t-b)}{q}}\right] dx$$

Load factor ($\frac{C_L}{C_h}$)	10% Threshold delays(R_hC_h)			50% Threshold delays (R_hC_h)		
	SPICE	**Rao95**	**Diff4**	**SPICE**	**Rao95**	**Diff4**
0.0	0.133	0.130	0.131	0.376	0.379	0.379
0.25	0.180	-	0.182	0.560	-	0.563
0.5	0.220	0.220	0.221	0.736	0.739	0.740
1.0	0.289	0.287	0.287	1.088	1.089	1.085
2.0	0.405	-	0.403	1.795	-	1.811
Load factor ($\frac{C_L}{C_h}$)	**63.2% Threshold delays(R_hC_h)**			**90% Threshold delays(R_hC_h)**		
	SPICE	**Rao95**	**Diff4**	**SPICE**	**Rao95**	**Diff4**
0.0	0.502	0.503	0.503	1.033	1.031	1.032
0.25	0.750	-	0.754	1.562	-	1.563
0.5	1.00	1.004	1.004	2.108	2.127	2.106
1.0	1.501	1.503	1.510	3.260	3.263	3.250
2.0	2.500	-	2.590	5.530	-	5.890

Table 1: A comparison of delay values at the end of the ($x = h$) between SPICE, the method of [14] and the analytical expression computed from diffusion analysis using up to first four reflection components.

with the function $F_2(s)$ being

$$\begin{aligned} F_2(s) &= -\frac{V_0}{s}\left(\frac{1-qs}{1+qs}\right)^2 e^{-bs} \\ &= -V_0\left[\frac{1}{s} - \frac{4/q}{(s+1/q)^2}\right] e^{-bs} \end{aligned}$$

where $b = \frac{(4h-x)}{h}\sqrt{R_hC_h}$. The function in time-domain can be obtained by taking the inverse transform as

$$f_2(t) = -V_0\left[1 - \frac{4(t-b)}{q}e^{-\frac{(t-b)}{q}}\right] U(t-b)$$

Substituting and making use of the identity in the Appendix, the time-domain expression for $v_{R_3}(x,t)$ is obtained as

$$\begin{aligned} v_{R_3}(x,t) &= \frac{1}{\sqrt{\pi t}}\int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} f_2(x)dx \\ &= \frac{-V_0}{\sqrt{\pi t}}\int_{x=b}^{\infty} e^{\frac{-x^2}{4t}}\left[1 - \frac{4(x-b)}{q}e^{-\frac{(x-b)}{q}}\right] dx \\ &= -V_0 erfc\left(\frac{b}{\sqrt{4t}}\right) + \frac{8V_0}{q}\sqrt{\frac{t}{\pi}}e^{-\left(\frac{b+2t/q}{\sqrt{4t}}\right)^2 + \frac{(t+qb)}{q^2}} - 4V_0\frac{(2t+qb)}{q^2}e^{\frac{(t+qb)}{q^2}} erfc\left(\frac{2t/q+b}{\sqrt{4t}}\right) \end{aligned}$$

where $U(t-b)$ is the step function which is 0 for $t < b$ and 1 for $t \geq b$. The inverse transform of this function with $\sqrt{s}$ as transform variable can be obtained using the identity (see the Appendix)

$$\frac{F_1(\sqrt{s})}{\sqrt{s}} \Longleftrightarrow \frac{1}{\sqrt{\pi t}} \int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} f_1(x) dx$$

Expressing the voltage response $V_{R_1}(x,s)$ in terms of the function F, we get

$$V_{R_1}(x,s) = \frac{F_1(\sqrt{s})}{\sqrt{s}}$$

and the corresponding time-domain response is[3]

$$\begin{aligned}
v_{R_1}(x,t) &= \frac{1}{\sqrt{\pi t}} \int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} V_0 (1 - 2e^{-\frac{(x-b)}{q}}) U(x-b) dx \\
&= V_0 erfc\left(\frac{b}{\sqrt{4t}}\right) - 2V_0 e^{\left(\frac{t+qb}{q^2}\right)} erfc\left(\frac{\sqrt{t}}{q} + \frac{b}{2\sqrt{t}}\right) \\
&= V_0 erfc\left(\frac{(2h-x)}{h}\sqrt{\frac{R_h C_h}{4t}}\right) \\
&\quad - 2V_0 e^{\left((\frac{C_h}{C_L})^2 \frac{t}{R_h C_h} + \frac{(2h-x)C_h}{h C_L}\right)} erfc\left(\frac{C_h}{C_L}\sqrt{\frac{t}{R_h C_h}} + \frac{(2h-x)}{2h}\sqrt{\frac{R_h C_h}{t}}\right) \\
&= V_0 erfc\left(\frac{(2h-x)}{h}\sqrt{\frac{R_h C_h}{4t}}\right) - 2V_0 e^{L1(C_L,1,1)} erfc(L2(C_L,1,1))
\end{aligned}$$

Similarly, the voltage response of the second reflected diffusion component (due to the source discontinuity) is given by

$$V_{R_2}(x,s) = \Gamma_S \Gamma_L V_I(2h+x,s) = -\Gamma_L V_I(2h+x,s)$$

and the corresponding time-domain response is

$$v_{R_2}(x,t) = -V_0 erfc\left(\frac{(2h+x)}{h}\sqrt{\frac{R_h C_h}{4t}}\right) + 2V_0 e^{L1(C_L,1,0)} erfc(L2(C_L,1,0))$$

Extending, we may calculate the voltage response of the third reflected diffusion component as

$$\begin{aligned}
V_{R_3}(x,s) &= \Gamma_S {\Gamma_L}^2 V_I(4h-x,s) \\
&= -\frac{V_0}{s}\left(\frac{1-q\sqrt{s}}{1+q\sqrt{s}}\right)^2 e^{-\frac{(4h-x)}{h}\sqrt{sR_h C_h}} = \frac{F_2(\sqrt{s})}{\sqrt{s}}
\end{aligned}$$

[3] Note that $\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = erfc(x)$.

components due to reflections at the load and source, and consists of infinite series of terms. It turns out that approximating the total response by considering up to the first four reflections is quite close to the SPICE-computed response. For larger loads than those we consider, more reflection components can be calculated by using numerical techniques.

In the following discussion we will use the following shorthand quantities, which can be used to express the time-domain voltage response in a simple and concise form. Let

$$\begin{aligned} L1(y,n,m) &= \left((\frac{C_h}{y})^2 \frac{t}{R_h C_h} + \frac{(2nh + (-1)^m x)}{h} \frac{C_h}{y}\right) \\ L2(y,n,m) &= \left(\frac{C_h}{y}\sqrt{\frac{t}{R_h C_h}} + \frac{(2nh + (-1)^m x)}{2h}\sqrt{\frac{R_h C_h}{t}}\right) \\ L3(y,n,m) &= \left((\frac{C_h}{y})^2 \frac{2t}{R_h C_h} + \frac{(2nh + (-1)^m x)}{h} \frac{C_h}{y}\right) \end{aligned} \tag{11}$$

where y can be either C_L or R_S, $n = 0, 1, 2, 3 \ldots$, and $m = 0$ for an even reflection component and $m = 1$ for an odd reflection component of the response.

The reflection coefficient at the load for an impedance of $Z_L = \frac{1}{sC_L}$ is

$$\Gamma_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1 - q\sqrt{s}}{1 + q\sqrt{s}} \tag{12}$$

where $q = C_L\sqrt{\frac{R_h}{C_h}} = \frac{C_L}{C_h}\sqrt{R_h C_h}$. From above, the reflection coefficient for an ideal source is $\Gamma_S(s) = \frac{Z_S - Z_0}{Z_S + Z_0} = -1$. The voltage response of the first reflected diffusion component at the load can be obtained from Equation (6):

$$\begin{aligned} V_{R_1}(x,s) &= \Gamma_L V_I(2h - x, s) \\ &= \frac{V_0}{s}\frac{1 - q\sqrt{s}}{1 + q\sqrt{s}} e^{-(2h-x)\sqrt{src}} = \frac{V_0}{s}\frac{1 - q\sqrt{s}}{1 + q\sqrt{s}} e^{-b\sqrt{s}} \end{aligned}$$

where $b = \frac{(2h-x)}{h}\sqrt{R_h C_h}$. To compute the time-domain response we apply the following technique to invert the above Laplace transform response. Let

$$F_1(s) = V_0 \frac{(1 - qs)}{s(1 + qs)} e^{-bs} = V_0 \left[\frac{1}{s} - \frac{2}{(s + 1/q)}\right] e^{-bs}$$

Applying the inverse transform yields

$$f_1(t) = V_0(1 - 2e^{-\frac{(t-b)}{a}})U(t - b)$$

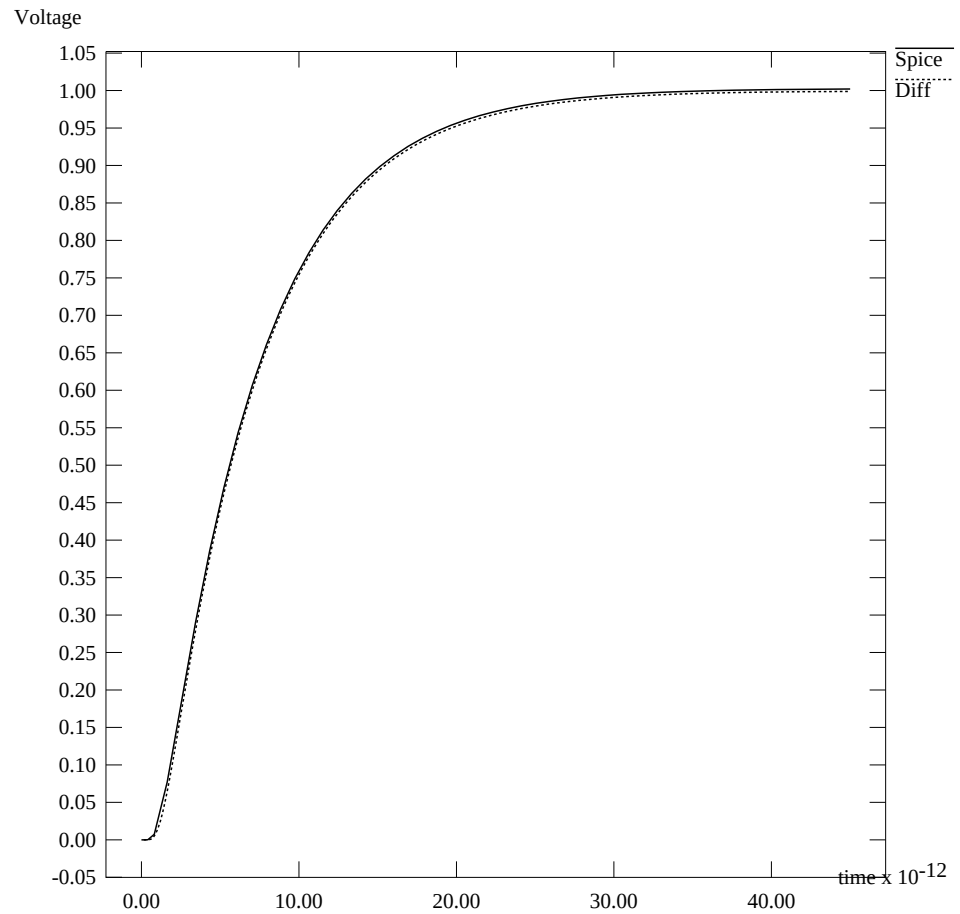

Figure 5: Voltage response for an open-ended finite RC line from SPICE and Diffusion approach. The time constant of the line is $R_h C_h = 15.0$ psec.

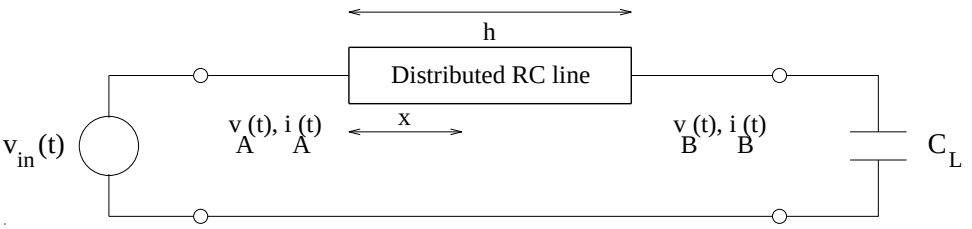

Figure 6: Distributed RC line of length h with capacitive load C_L.

We now consider a finite distributed RC line of length h with capacitive load C_L at the end of the line (Figure 6). This case has been previously treated in [14] via an extension of the classical transform-based analysis; the time-domain response was then calculated by considering the first 10 dominant poles.

From Equation (8), the total response on the line is obtained by adding up all the voltage

the i^{th} reflection for odd $i = 2n-1$ is

$$\begin{aligned} V_{R_i}(x,s) &= \Gamma_L^{\frac{i+1}{2}}(s)\Gamma_S^{\frac{i-1}{2}}V_I(h(i+1)-x,s) = (-1)^{\frac{i-1}{2}}V_I(h(i+1)-x,s) \\ &= (-1)^{n-1}V_I(2nh-x,s), \end{aligned}$$

and the respective time-domain responses are

$$v_{R_n}(x,t) = \begin{cases} (-1)^n v_I(2nh+x,t) = (-1)^n erfc\left(\sqrt{\frac{R_{2nh+x}C_{2nh+x}}{4t}}\right) & \text{for even } n \\ (-1)^{n-1} v_I(2nh-x,t) = (-1)^{n-1} erfc\left(\sqrt{\frac{R_{2nh-x}C_{2nh-x}}{4t}}\right) & \text{for odd } n \end{cases}$$

Therefore, the total voltage is

$$\begin{aligned} v_{Tot}(x,t) &= v_I(x,t) + \sum_{n=1}^{\infty}\left[(-1)^n v_I(2nh+x)(x,t) + (-1)^{n-1}v_I(2nh-x,t)\right] \\ &= V_0 erfc\left(\sqrt{\frac{R_x C_x}{4t}}\right) \\ &\quad + V_0\sum_{n=1}^{\infty}\left[(-1)^n erfc\left(\sqrt{\frac{R_{2nh+x}C_{2nh+x}}{4t}}\right) + (-1)^{n-1}erfc\left(\sqrt{\frac{R_{2nh-x}C_{2nh-x}}{4t}}\right)\right] \end{aligned} \tag{10}$$

This expression has also been derived from the transform-domain approach in [8, 14]. For $x = h$ the distributed RC line response reduces to that given in [11], i.e.,

$$v_{Tot}(h,t) = 2V_0\sum_{n=0}^{\infty}(-1)^n erfc\left(\sqrt{\frac{R_{(2n+1)h}C_{(2n+1)h}}{4t}}\right) = 2V_0\sum_{n=0}^{\infty}(-1)^n erfc\left((2n+1)\sqrt{\frac{R_h C_h}{4t}}\right)$$

It is easy to verify that this expression for total voltage satisfies the boundary condition at the end of the line, i.e., Equation (9), since

$$\begin{aligned} i(h,t) &= \lim_{x\to h}\frac{-1}{r}\frac{\partial v_{Tot}(x,t)}{\partial x} \\ &= V_0\sqrt{\frac{rc}{t\pi}}e^{\frac{rch^2}{4t}} + V_0\sqrt{\frac{rc}{t\pi}}\sum_{n=1}^{\infty}\left[(-1)^n e^{\frac{rch^2(2n+1)^2}{4t}} + (-1)^n e^{\frac{rch^2(2n-1)^2}{4t}}\right] \end{aligned}$$

All the terms in the above equation cancel so that $i(h,t) = 0$. Figure 5 shows that the voltage responses computed from SPICE and from the diffusion equation approach are very close.

4 Finite RC Line with Capacitive Load

For a finite-length transmission line, the total voltage should satisfy the boundary condition at the end of the line, which is

$$i(h,t) = \lim_{x \to h} \frac{-1}{r} \frac{\partial v_{Tot}(x,t)}{\partial x} \tag{9}$$

Also, the current at the end of the line is the same as the load current, i.e.,

$$I(h,s) = \frac{V_{Tot}(h,s)}{Z_L}$$

Hence, the time-domain current can be obtained for any specific case by taking the inverse Laplace transform of $I(h,s)$.

3 The Open-ended Finite RC Line

The above analysis can be used to correctly compute the time-domain response for the open-ended RC line; the result indeed coincides with the classical result obtained with the transform domain method [11, 8, 14]. The finite-length, open-ended RC transmission line with ideal source has $Z_S = 0$ (i.e., $\Gamma_S = -1$ and $\Gamma_S(t) = \delta(t)$), and $Z_L = \infty$ (i.e., $\Gamma_L(s) = 1$ and $\Gamma_L(t) = \delta(t)$). The first reflected voltage is

$$V_{R_1}(x,s) = \Gamma_L(s) V_I(2h - x, s) = V_I(2h - x, s)$$

and the corresponding time-domain response is

$$V_{R_1}(x,t) = V_I(2h - x, t) = V_0 erfc\left(\sqrt{\frac{R_{2h-x}C_{2h-x}}{4t}}\right)$$

Similarly, the second reflected voltage is

$$V_{R_2}(x,s) = \Gamma_L(s)\Gamma_S(s) V_I(2h - x, s) = -V_I(2h + x, s)$$

with

$$V_{R_2}(x,t) = -V_I(2h + x, t) = -V_0 erfc\left(\sqrt{\frac{R_{2h+x}C_{2h+x}}{4t}}\right)$$

In general, the i^{th} reflection for even $i = 2n$ in the transform domain is

$$\begin{aligned} V_{R_i}(x,s) &= \Gamma_L^{\frac{i}{2}}(s)\Gamma_S^{\frac{i}{2}}(s) V_I(ih + x, s) = (-1)^{\frac{i}{2}} V_I(ih + x, s) \\ &= (-1)^n V_I(2nh + x, s), \end{aligned}$$

in position by $h + h - x = 2h - x$ and is given by

$$V_{R_1}(x,s) = \Gamma_L(s) V_I(2h - x, s). \tag{6}$$

The corresponding time-domain expression is

$$v_{R_1}(x,t) = \int_{\tau=0}^{t} \Gamma_L(t-\tau) v_I(2h - x, \tau)\, d\tau,$$

i.e., the first reflected voltage travels a distance h to the end of the line before reflection and then $h - x$ to reach the desired location. The analysis of [6, 7] neglected to shift in position and thus incorrectly obtained cancellations of reflected components in the computation of the response for open-ended line case. Another explanation for the reflection voltages is by applying the symmetry argument in the Method of Images (or Reflections) [10] to satisfy the boundary condition at the end of the line $x = h$, as shown in Figure 4. in Section 6, we also present another approach, using the 2-port transfer function of the line, for calculating the total response on the line and show that total voltage can be expressed as a sum of various reflected components.

Similarly, the second reflection at the source yields $V_{R_2}(x,s) = \Gamma_S(s)\Gamma_L(s) V_I(2h + x, s)$, and in general the i^{th} reflection gives

$$V_{R_i}(x,s) = \begin{cases} \Gamma_L^{\frac{i}{2}}(s)\Gamma_S^{\frac{i}{2}}(s) V_I(ih + x, s) & \text{for } i \text{ even} \\ \Gamma_L^{\frac{i+1}{2}}(s)\Gamma_S^{\frac{i-1}{2}} V_I(h(i+1) - x, s) & \text{for } i \text{ odd} \end{cases}$$

Substituting $i = 2n$ for even i, and $i = 2n - 1$ for odd i, we get

$$\begin{aligned} V_{Tot}(x,s) &= V_I(x,s) + \sum_{i=1}^{\infty} V_{R_i}(x,s) \\ &= V_I(x,s) + \sum_{n=1}^{\infty} \left(\Gamma_L^n(s)\Gamma_S^{n-1} V_I(2nh - x, s) + \Gamma_L^n(s)\Gamma_S^n(s) V_I(2nh + x, s) \right) \end{aligned} \tag{7}$$

and the time-domain response is

$$\begin{aligned} v_{Tot}(x,t) &= v_I(x,t) + \sum_{n=1}^{\infty} \left[a_n(t) \otimes v_I(2nh - x, t) + b_n(t) \otimes v_I(2nh + x, t) \right] \\ &= v_I(x,t) + \sum_{n=1}^{\infty} \left[\int_{\tau=0}^{t} a_n(t-\tau) v_I(2nh - x, \tau)\, d\tau + \int_{\tau=0}^{t} b_n(t-\tau) v_I(2nh + x, \tau)\, d\tau \right] \end{aligned} \tag{8}$$

where $a_n(t)$ and $b_n(t)$ represent odd and even n^{th} reflection coefficient values. The total voltage at the end of the transmission line of length h is

$$v_{Tot}(h,t) = v_I(h,t) + \sum_{n=1}^{\infty} \left[\int_{\tau=0}^{t} a_n(t-\tau) v_I(h(2n-1), \tau)\, d\tau + \int_{\tau=0}^{t} b_n(t-\tau) v_I(h(2n+1), \tau)\, d\tau \right]$$

where $v_I(x,t) \equiv$ voltage due to the incident diffusion and $v_{R_i}(x,t) \equiv$ voltage due to the i^{th} reflection.[2] The expression for any $v_{R_i}(x,t)$ will be of the same form as the expression for the incident voltage $v_I(x,t)$, but with a displacement in the position variable and with a different magnitude (according to the reflection coefficients). In general, $v_{R_i}(x,t)$ can be calculated through convolution of the reflected diffusion taking into account position displacement, with the reflection coefficients $\Gamma_S(t)$ or $\Gamma_L(t)$. (Note that, e.g., $\Gamma_S(t) = \mathcal{L}^{-1}\{\Gamma_S(s)\}$.) Alternatively, one can take the inverse Laplace transform of $v_{R_i}(x,s)$.

General Case: Finite-length RC transmission line with general source and load impedances Z_S and Z_L.

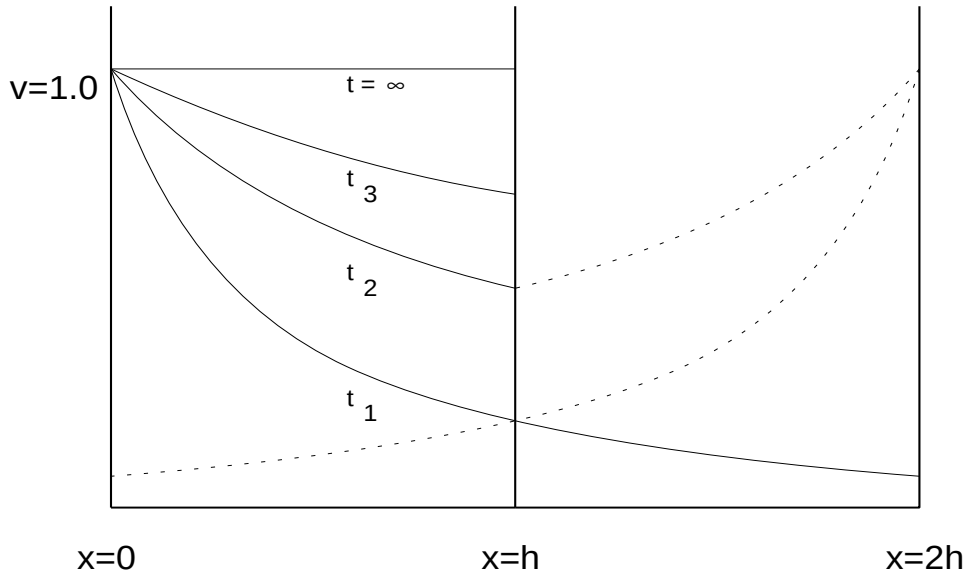

Figure 4: The evolution of the voltage profile for a finite distributed RC line of length h. The dotted line indicates the voltage due to the first reflection at $x = h$, which can also be obtained from the Method of Images (or Reflections) [12].

The reflection coefficient at the source in the transform domain is $\Gamma_S(s) = \frac{Z_S - Z_0}{Z_S + Z_0}$, and the reflection coefficient at the load is $\Gamma_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0}$. The voltage at the position x (Figure 3) due to the first reflection at the load can be calculated by considering the incident wave and shifting

[2]Similarly, the total voltage in transform domain is $V_{Tot}(x,s) = V_I(x,s) + \sum_{i=1}^{\infty} V_{R_i}(x,s)$. In our notation, R_i refers to the i^{th} *reflected diffusion* starting from either the source or the load discontinuity; i basically represents the number of trips up and down the line. Finally, observe that Equation (3) also applies for finite-length distributed RC line with perfectly matched load, since there is no reflection at the load, i.e., only the incident diffusion voltage along the line.

and the transmission coefficient is $(1 + \Gamma)$. The concepts of reflected and transmitted voltages apply at any discontinuity; in particular, we may use these reflection coefficients to calculate voltage propagation over a finite-length distributed RC line.

2.3 The Finite Distributed RC Line

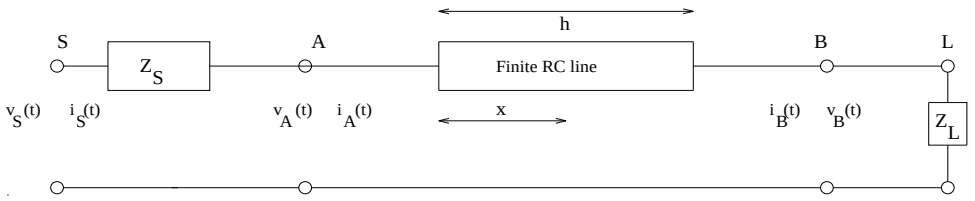

Figure 3: A distributed RC line of length h, and position x along the line with source impedance Z_S and load impedanced Z_L.

For a finite RC line with ideal source, the diffusion equation solution given in Equation (3) can be viewed as an *incident* propagation of voltage due to a step input. In general for a finite RC line with source and load impedance as shown in Figure 3, the incident propagation of the voltage on the line in transform domain can be obtained from Equation (4) as

$$V_I(x, s) = V_A(s)e^{-\sqrt{R_x C_x s}}$$

where $R_x = xr$, $C_x = xc$ and r, c are resistance and capacitance per unit length. The voltage at the front end of the line (i.e, at A) for a step input is

$$\begin{aligned} V_A(s) &= \frac{V_0}{s} \cdot \frac{Z_0}{Z_0 + Z_S} \\ &= \frac{V_0(1 - \Gamma_S(s))}{2s} \end{aligned}$$

Therefore,

$$V_I(x, s) = \frac{V_0(1 - \Gamma_S(s))}{2s} e^{-\sqrt{R_x C_x s}}$$

The time-domain expression of the incident voltage can be obtained by taking inverse Laplace transform and is indicated by $v_I(x, t)$.

The total voltage on the line (Figure 3) is the summation of the incident diffusion component plus *reflected diffusion components* due to discontinuities at the source (S) and load (L). In other words, the time-domain expansion for total voltage is

$$v_{Tot}(x, t) = v_I(x, t) + \sum_{i=1}^{\infty} v_{R_i}(x, t) \tag{5}$$

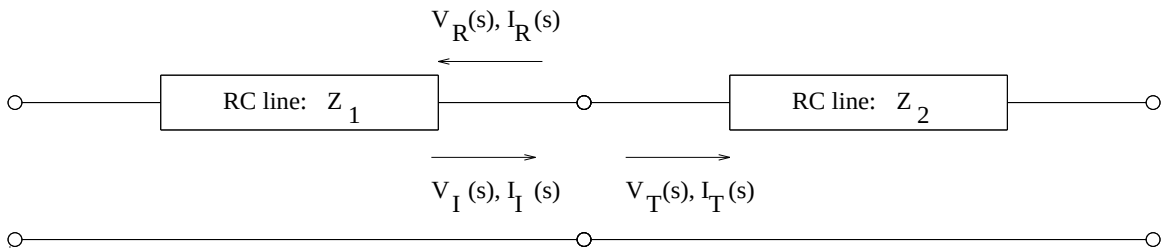

Figure 2: Connection of two RC lines showing incident, reflected and transmitted components at the junction.

to a resistive driver. These mismatches of impedance cause the voltage on the line to change according to proper reflections [2].

Consider the arrival of an incident voltage and current $(V_I(s), I_I(s))$ at a junction between two distributed RC lines of different impedances, as shown in Figure 2. Portions of this voltage and current $(V_T(s), I_T(s))$ get transmitted to the second line, while $(V_R(s), I_R(s))$ are reflected back to the first line. From continuity of voltage across the junction,

$$V_I(s) + V_R(s) = V_R(s)$$

and conservation of charge yields

$$I_I(s) + I_R(s) = I_T(s)$$

The incident, the transmitted, and the reflected voltages satisfy the conditions

$$V_I(s) = Z_1 I_I(s) \qquad V_T(s) = Z_2 I_T(s) \qquad V_R(s) = Z_1 I_R(s)$$

where Z_1 and Z_2 are the characteristic impedances of the first and second lines. Using the above equations and solving for the transmitted and the reflected voltages in terms of the incident voltage, we have

$$V_T(s) = V_I(s)\frac{2Z_2}{Z_2 + Z_1}$$
$$V_R(s) = V_I(s)\frac{Z_2 - Z_1}{Z_2 + Z_1}$$

From the above equations, the reflection coefficient at the discontinuity is defined as

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

[1] The error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ and the complementary error function is defined as $erfc(x) = 1 - erf(x)$.

which can be solved by restricting to solutions of form $v(\frac{x}{\sqrt{t}})$ using the substitution variable $\eta = x\sqrt{\frac{rc}{2t}}$. The general solution is [6, 7]

$$v(\eta) = C_1 \int_0^{\eta} e^{-\frac{\eta^2}{2}} d\eta + C_2 \tag{1}$$

The initial and boundary conditions for the semi-infinite line are

$$\begin{array}{lrcll} \text{IC:} & v(x,0) & = & 0 & \text{for all } x \geq 0 \\ \text{BC1:} & v(0,t) & = & v_A(t) & \text{for all } t \geq 0 \end{array} \tag{2}$$

and the initial condition that the line is quiet at $t = 0$ implies $C_1 = -\sqrt{\frac{2}{\pi}} C_2$.

With an ideal source and a step input ($V_{in}(s) = \frac{V_0}{s}$), the voltage at the front end of the line is $v(0,t) = v_{in}(t) = V_0 u(t)$, i.e., at position $x = 0$ the voltage is constant and equal to V_0 for all $t > 0$. Using this condition in Equation (1) yields

$$V_0 = C_1 \int_0^{\eta = x\sqrt{\frac{rc}{2t}} = 0} e^{-\frac{x^2}{2}} dx + C_2$$

from which $C_2 = V_0$ and[1]

$$v_I(x,t) = v(\eta) = V_0[1 - erf(\frac{\eta}{\sqrt{2}})] = V_0 \; erfc\left(x\sqrt{\frac{rc}{4t}}\right). \tag{3}$$

Later, it will be useful to note that the transform domain analog [4] of Equation (3) is

$$V_I(x,s) = \frac{V_0}{s} \cdot e^{-x\sqrt{rcs}}$$

and that for a general input at the front end of the line $V_A(s)$ the response is

$$V_I(x,s) = V_A(s) \cdot e^{-x\sqrt{rcs}} = V_A(s) \cdot H_I(s) \tag{4}$$

where $H_I(s)$ corresponds to the transfer function of the semi-infinite line.

2.2 Analysis of Discontinuities

An actual distributed transmission line is finite in length, implying a change of impedance at the end of the line. The discontinuity at the end of the line could be due to another distributed line of different impedance or due to a load impedance. For typical VLSI interconnections, discontinuities at the end of the line are due to capacitive load, and those at the source are due

the recent method of [14]. For higher accuracy in the response, e.g., for larger loads, additional terms can be computed using numerical integration methods. Our analysis of RC lines can be extended to response computations in general interconnection trees, by considering both reflection and transmission components at discontinuities (see Section 2.2).

To complete the discussion, we also derive the voltage response from the classical 2-port transfer function of the distributed line and show that it exactly matches the response derived using the diffusion approach.

2 Diffusion Based Analysis of the RC Line

2.1 Semi-Infinite Line Analysis

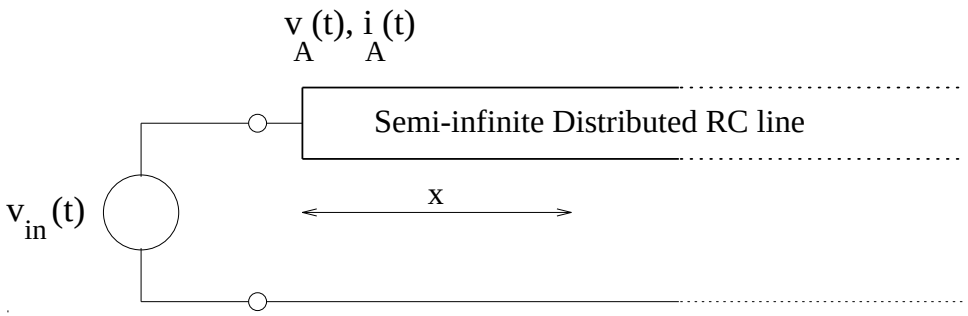

Figure 1: A semi-infinite distributed RC line, and position x along the line.

Consider the semi-infinite distributed RC line shown in Figure 1. The PDEs governing the distributed RC line are

$$\begin{aligned} \frac{\partial i(x,t)}{\partial x} &= -c(x)\frac{\partial v(x,t)}{\partial t} \\ \frac{\partial v(x,t)}{\partial x} &= -r(x)i(x,t) \end{aligned}$$

where $r(x)$ and $c(x)$ are resistance and capacitance per unit length. In a uniform wire, $r(x) = r$ and $c(x) = c$ are constants and we obtain the diffusion equation

$$rc\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

circuit or perfectly matched load [13, 16, 1, 12]. Using a different approach to invert the Laplace transform of the response, [11, 8, 14] have all obtained the exact time-domain response for a finite-length open-ended RC line. The most recent of these works, by Rao [14], also extends the traditional transform-domain analysis to calculate the approximate time-domain response for a finite RC line with capacitive load impedance. The time-domain response is approximated by calculating 10 dominant poles and their corresponding residues from the transform-domain response; pole values are calculated by solving the transcendental equation numerically.

A direct solution of the open-ended finite RC line response, i.e., directly in the time-domain, was first given by Kaufman and Garrett [9]. To obtain the transient response to a step input, [9] makes the simplifying assumption $v(x,t) = f(x) \cdot g(t)$, i.e., that the voltage response is separable into functions of position and time. An infinite series is obtained, and the response can be approximated by considering as many dominant poles as needed. More recently, Kahng and Muddu [6, 7] also gave a direct time-domain analysis of voltage response: in a finite distributed RC line, the total response was shown to be equal to the infinite sum of diffusion equation solutions, with each diffusion starting at either end of the line. Various cases of source and load impedances either infinity or zero were discussed, e.g., the open-ended line, short-circuit load, and perfectly matched load. However, analysis of the open-ended line case was incorrect; [14] has pointed out that the response given is actually that of the semi-infinite line.

In this paper, we make the following contributions.

1. We begin with the solution of diffusion in a semi-infinite distributed RC line [7, 14] and apply the method of images (reflections) to obtain the correct voltage response for an open-ended finite RC line. This corrects the erroneous analysis of [6, 7].

2. We extend this to a general approach for calculating the response for an RC line with general source and load impedances. The total response is obtained by adding various individual diffusion components; each of these can be viewed as traveling either forward or backward along the line, in analogy with reflections.

3. We obtain new approximate analytical expressions for the voltage response in a finite RC line with resistive source and capacitive load. Delay estimates are very close to SPICE-computed delays; we give comparisons with both SPICE and, for the capacitive load case,

A New Reflection-Based Approach for *RC* Interconnect Analysis *

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Abstract

This paper gives new methods for calculating the time-domain response for a finite-length distributed *RC* line. The following are our contributions. First, we begin with the solution of diffusion in a semi-infinite distributed *RC* line [7, 14] and apply the method of images (reflections) to obtain the voltage response for a *finite-length RC* line. Second, we describe a general approach for calculating the time-domain response for finite *RC* lines with source and load impedances. We then obtain new approximate analytical expressions for the voltage response in a finite *RC* line with resistive source and capacitive load. The delay estimates using this method are very close to SPICE-computed delays; we compare against both SPICE and the recent method of [14] which approximates delays in an *RC* line with capacitive load. Our method is simple; for higher accuracy in the response, additional terms can be computed using numerical techniques.

1 Introduction

Analysis of on-chip interconnections is a major concern for high-performance VLSI designs, since resistive and capacitive effects of wires increase as chip sizes grow larger and feature sizes are reduced [5]. Estimating delays on VLSI interconnections is a key element in timing verification, gate-level simulation and performance-driven layout design. Because of the highly resistive nature of the wires, many tools model the wires inside current integrated circuits as distributed *RC* lines.

The analysis of open-ended finite *RC* transmission lines is widely discussed in the literature, e.g., [13, 16, 15]. The standard approach is to first calculate the transfer function; then, by approximating the transfer function both transform-domain and time-domain responses are obtained. This approach can extend to calculate the response for such simple cases as a short

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