Two New Reformulations of Gaussian Elimination

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Abstract

In this paper we use Sylvester's determinantal identity to derive two reformulations of the Gaussian Elimination algorithm. One reformulation obviates the need for division operations, and another produces intermediate results that are always specific determinants of submatrices of the input matrix. Both algorithms find $L\,D^{-1}\,U$ decompositions of the input matrix A, and $L\,D^{-1}\,L^{\top}$ decompositions when A is symmetric. Although they perform quite different computations than the traditional algorithm, they differ formally only by an underlying change of variables.

1 Introduction

Today high-performance computers are used heavily to solve large linear systems, and they run variants of Gaussian Elimination and LU decomposition constantly [2, 8]. Gaussian Elimination is a prominent topic in every text on computational linear algebra. A comprehensive overview of modern numerical software references up to the late-1980's can be found in Chapter 3 of [5].

Gaussian Elimination is often presented as the **for**–loop program shown in Figure 1, requiring $n(n^2-1)/3$ assignments altogether. In this form, Gaussian Elimination performs a sequence of transformations to a $n \times n$ square matrix $A = (a_{ij})$, reducing it to upper-diagonal form in n steps.

for $k = 1$ **to** $n-1$ **do**
  for $i = k+1$ **to** n **do**
    begin
      $\ell_{ik} \ := \ a_{ik} \,/\, a_{kk};$ $\qquad \{ \ a_{ik}^{(k)} \,/\, a_{kk}^{(k)} \ \}$
      for $j = k+1$ **to** n **do**
        $a_{ij} \ := \ a_{ij} \ - \ \ell_{ik} \ \cdot \ a_{kj}$ $\qquad \{ \ a_{ij}^{(k)} \ - \ a_{ik}^{(k)} \,/\, a_{kk}^{(k)} \ \cdot \ a_{kj}^{(k)} \ \}$
    end

Figure 1: Standard Gaussian Elimination

Gaussian elimination can be defined equationally, with the initial assignment $a_{ij}^{(1)} = a_{ij}$ and the recursive definition of $a_{ij}^{(k+1)}$ for $1 \leq k \leq n-1$:

$$a_{ij}^{(k+1)} = \begin{cases} 0 & i \geq k+1,\ j = k \\ a_{ij}^{(k)} - a_{ik}^{(k)} / a_{kk}^{(k)} \cdot a_{kj}^{(k)} & i \geq k+1,\ j \geq k+1 \\ a_{ij}^{(k)} & \text{otherwise.} \end{cases}$$

Starting with $A^{(1)} = A$, each iteration of the outer loop computes $A^{(k+1)} = (a_{ij}^{(k+1)})$ for $1 \leq k \leq n-1$, except that as written above the program does not zero the elements of A below the diagonal in the first k columns. (These elements are in precisely the same positions as the multipliers ℓ_{ik}, so many implementations store ℓ_{ik} in a_{ik}.) With the program we can form the triangular matrices

$$L = \begin{pmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} & a_{13}^{(n)} & \cdots & a_{1n}^{(n)} \\ & a_{22}^{(n)} & a_{23}^{(n)} & \cdots & a_{2n}^{(n)} \\ & & a_{33}^{(n)} & \cdots & a_{3n}^{(n)} \\ & & & \ddots & \vdots \\ & & & & a_{nn}^{(n)} \end{pmatrix}$$

such that $A^{(n)} = U$ and $A = LU$, provided such a decomposition exists. The computation reduces a general linear problem of the form $A\mathbf{x} = \mathbf{b}$ to a pair of back-substitution problems $L\mathbf{y} = \mathbf{b}$, $U\mathbf{x} = \mathbf{y}$. For a particularly good introduction see [4]. It is well known that A has an LU decomposition, and Gaussian Elimination succeeds in finding it, iff the $a_{kk}^{(k)}$ are nonzero, for all k:

Definition 1 *An $n \times n$ matrix A is* **Gauss eliminable** *if $a_{kk}^{(k)} \neq 0$, for $1 \leq k \leq n$.*

For clarity, throughout this paper we will consider only Gauss eliminable matrices.

Recently we derived a number of properties of Gaussian elimination [7], and as a result realized there are new ways to reformulate the entire algorithm. In this paper we show that, assuming A is Gauss eliminable, the algorithm above yields the same results as two reformulated algorithms. This equivalence is surprising. For example, when exact computation is used, standard Gaussian Elimination yields the same results as the algorithm shown in Figure 2. Moreover, with only the first **for**-block of this reformulation, it is possible to directly obtain an $L\,D^{-1}\,U$ decomposition.

for $k = 1$ **to** $n-1$ **do**
  for $i = k+1$ **to** n **do**
    for $j = k+1$ **to** n **do**
      $a_{ij} := (\, a_{ij} \cdot a_{kk} - a_{ik} \cdot a_{kj}\,) \,/\, a_{k-1,k-1};$ { compute $\beta_{ij}^{(k+1)}$ }

for $i = 1$ **to** n **do** **for** $k = 1$ **to** $i-1$ **do** $\ell_{ik} := a_{ik} \,/\, a_{kk}$; { compute L }

for $k = n$ **by** -1 **to** 2 **do** **for** $j = k$ **to** n **do** $a_{kj} := a_{kj} \,/\, a_{k-1,k-1}$ { compute U }

Figure 2: A New Reformulation of Gaussian Elimination

2 A Division-Free Reformulation of Gaussian Elimination

We show first how a reformulated algorithm, yielding an $LD^{-1}U$ decomposition with no division operations, is obtainable directly from Gaussian Elimination. Consider the change of variables

$$\alpha_{ij}^{(k)} \quad = \quad a_{ij}^{(k)} \; d_{k-1}$$

for $1 \leq k < i \leq n$, $1 \leq k < j \leq n$, where we define $d_0 = 1$ (so $\alpha_{ij}^{(1)} = a_{ij}^{(1)} d_0 = a_{ij}$), and

$$d_k \quad = \quad \prod_{\kappa=1}^{k} \alpha_{\kappa\kappa}^{(\kappa)}$$

for $1 \leq k \leq n$, so in general $d_k = d_{k-1} \cdot a_{kk}^{(k)}$. This change of variables yields a division-free iteration:

Theorem 1 *For $1 \leq k < i, j \leq n$,* $\boxed{\alpha_{ij}^{(k+1)} \quad = \quad \alpha_{ij}^{(k)} \cdot \alpha_{kk}^{(k)} - \alpha_{ik}^{(k)} \cdot \alpha_{kj}^{(k)}}$.

Proof

$$\begin{aligned}
\alpha_{ij}^{(k+1)} &= a_{ij}^{(k+1)} \cdot d_k \\
&= \left(a_{ij}^{(k)} - \frac{a_{ik}^{(k)} \cdot a_{kj}^{(k)}}{a_{kk}^{(k)}} \right) \cdot d_k \\
&= \left(\frac{\alpha_{ij}^{(k)}}{d_{k-1}} - \frac{\alpha_{ik}^{(k)}}{d_{k-1}} \cdot \frac{d_{k-1}}{\alpha_{kk}^{(k)}} \cdot \frac{\alpha_{kj}^{(k)}}{d_{k-1}} \right) \cdot d_k \\
&= \left(\frac{\alpha_{ij}^{(k)} \cdot \alpha_{kk}^{(k)} - \alpha_{ik}^{(k)} \cdot \alpha_{kj}^{(k)}}{d_{k-1} \cdot \alpha_{kk}^{(k)}} \right) \cdot d_k \\
&= \alpha_{ij}^{(k)} \cdot \alpha_{kk}^{(k)} - \alpha_{ik}^{(k)} \cdot \alpha_{kj}^{(k)} .
\end{aligned}$$

□

Assuming A is Gauss eliminable, the standard Gaussian Elimination LU decomposition algorithm can be shown to formally compute the same results as the reformulated algorithm shown in Figure 3.

We reformulate Gaussian Elimination by iteratively computing $\alpha_{ij}^{(k+1)}$, and afterwards converting the results to the desired $a_{ij}^{(k+1)}$ values. In order to get a program that computes the same results as the standard one, we co-locate the variables $\alpha_{ij}^{(k)}$ with a_{ij} as it does, which works well since $\alpha_{ij}^{(1)} = a_{ij}$. The final **for**-loop of the program computes the $u_{ij} = a_{ij}^{(n)} = a_{ij}^{(i)}$ values, when $i < j$.

Theorem 2 *Assuming A is Gauss eliminable, standard Gaussian Elimination and the program in Figure 3 formally compute the same results.*

Proof is easily done by induction on k, requiring only that we show:

$$\begin{aligned}
\ell_{ik} &= \alpha_{ik}^{(k)} \;/\; \alpha_{kk}^{(k)} \\
a_{ij}^{(k)} &= \alpha_{ij}^{(k)} \;/\; d_{k-1}.
\end{aligned}$$

for $k = 1$ **to** $n-1$ **do**
  for $i = k+1$ **to** n **do**
    for $j = k+1$ **to** n **do**
      $a_{ij} \;:=\; (a_{ij} \cdot a_{kk} \;-\; a_{ik} \cdot a_{kj})\;;$ $\{$ compute $\alpha_{ij}^{(k+1)}$ $\}$

$d_0 \;:=\; 1;$ **for** $k = 1$ **to** n **do** $d_k \;:=\; d_{k-1} \cdot a_{kk}\;;$

for $i = 1$ **to** n **do** **for** $k = 1$ **to** $i-1$ **do** $\ell_{ik} \;:=\; a_{ik} \;/\; a_{kk}\;;$ $\{$ compute L $\}$

for $i = 1$ **to** n **do** **for** $j = i$ **to** n **do** $a_{ij} \;:=\; a_{ij} \;/\; d_{i-1}$ $\{$ compute U $\}$

Figure 3: Another Reformulation of Gaussian Elimination

Both these equations follow from the change of variables definition above. □

This reformulation also permits an $LD^{-1}U$ decomposition for A that can be computed with no use of division (where neither L nor U is unit-triangular). Specifically:

Theorem 3 *Any Gauss-eliminable matrix A has the decomposition*

$$A \;=\; \begin{pmatrix} \alpha_{11}^{(1)} & & & & \\ \alpha_{21}^{(1)} & \alpha_{22}^{(2)} & & & \\ \alpha_{31}^{(1)} & \alpha_{32}^{(2)} & \alpha_{33}^{(3)} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \alpha_{n1}^{(1)} & \alpha_{n2}^{(2)} & \alpha_{n3}^{(3)} & \cdots & \alpha_{nn}^{(n)} \end{pmatrix} \begin{pmatrix} 1/d_1 & & & & \\ & 1/d_2 & & & \\ & & 1/d_3 & & \\ & & & \ddots & \\ & & & & 1/d_n \end{pmatrix} \begin{pmatrix} \alpha_{11}^{(1)} & \alpha_{12}^{(1)} & \alpha_{11}^{(1)} & \cdots & \alpha_{1n}^{(1)} \\ & \alpha_{22}^{(2)} & \alpha_{23}^{(2)} & \cdots & \alpha_{2n}^{(2)} \\ & & \alpha_{33}^{(3)} & \cdots & \alpha_{3n}^{(3)} \\ & & & \ddots & \vdots \\ & & & & \alpha_{nn}^{(n)} \end{pmatrix}$$

Proof

This follows from the LU decomposition for standard Gaussian Elimination

$$A \;=\; L\,U \;=\; \begin{pmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} & a_{11}^{(n)} & \cdots & a_{1n}^{(n)} \\ & a_{22}^{(n)} & a_{23}^{(n)} & \cdots & a_{2n}^{(n)} \\ & & a_{33}^{(n)} & \cdots & a_{3n}^{(n)} \\ & & & \ddots & \vdots \\ & & & & a_{nn}^{(n)} \end{pmatrix}$$

because the equations $\ell_{ik} = \alpha_{ik}^{(k)}/\alpha_{kk}^{(k)}$, $a_{ij}^{(k)} = \alpha_{ij}^{(k)}/d_{k-1}$ permit reorganizing the decomposition as $A \;=\; L_\alpha\, D_\alpha^{-1}\, U_\alpha$ where (in keeping with the notation in Figure 3)

$$L_\alpha \;=\; L \;\; diag(\alpha_{kk}^{(k)}) \;=\; (\alpha_{ik}^{(k)}) \qquad \text{(where } i \geq k\text{: } L_\alpha \text{ is lower triangular)}$$

$$D_\alpha \;=\; \Big(diag(\alpha_{kk}^{(k)})^{-1} \;\; diag(d_{k-1})^{-1}\Big)^{-1} \;=\; diag(d_{k-1}\,\alpha_{kk}^{(k)}) \;=\; diag(d_k)$$

$$U_\alpha \;=\; diag(d_{k-1}) \;\; U \;=\; (\alpha_{ij}^{(k)}) \qquad \text{(where } i \leq j\text{: } D_\alpha \text{ is upper triangular).}$$

□

Note that the diagonals of L_α and U_α are identical, and *not* usually unit-valued. Also, when A is symmetric, we obtain $U_\alpha = {L_\alpha}^\top$; thus we could speed up the algorithm for symmetric matrices by computing only half of the α values.

This result is intriguing since all numerical inversion (division) is performed in computing $D_\alpha^{-1} \;=\; diag(d_k)^{-1}$. Non-Gauss-eliminable matrices are thus manifested by diagonal matrices $D_\alpha = diag(d_k)$ with zero elements.

3 Another Reformulation of Gaussian Elimination

We now show how the reformulated algorithm shown earlier in Figure 2 is obtainable from Gaussian Elimination. Consider the change of variables for $1 \leq k < i \leq n$, $1 \leq k < j \leq n$,

$$\beta_{ij}^{(k)} \quad = \quad a_{ij}^{(k)} \;\cdot\; \beta_{k-1,k-1}^{(k-1)}$$

where we define $\beta_{00}^{(0)} = 1$. This change of variables also yields a simple iteration:

Theorem 4 *For* $1 \leq k < i,\, j \leq n$, $\boxed{\beta_{ij}^{(k+1)} \quad = \quad \left(\beta_{ij}^{(k)} \cdot \beta_{kk}^{(k)} \;-\; \beta_{ik}^{(k)} \cdot \beta_{kj}^{(k)}\right) \;/\; \beta_{k-1,k-1}^{(k-1)}}$.

Proof

$$\begin{aligned}
\beta_{ij}^{(k+1)} &= a_{ij}^{(k+1)} \cdot \beta_{kk}^{(k)} \\
&= \left(a_{ij}^{(k)} \;-\; a_{ik}^{(k)} \cdot \frac{1}{a_{kk}^{(k)}} \cdot a_{kj}^{(k)} \right) \cdot \beta_{kk}^{(k)} \\
&= \left(\beta_{ij}^{(k)} / \beta_{k-1,k-1}^{(k-1)} \;-\; \beta_{ik}^{(k)} / \beta_{k-1,k-1}^{(k-1)} \cdot \frac{1}{\beta_{kk}^{(k)} / \beta_{k-1,k-1}^{(k-1)}} \cdot \beta_{kj}^{(k)} / \beta_{k-1,k-1}^{(k-1)} \right) \cdot \beta_{kk}^{(k)} \\
&= \left(\beta_{ij}^{(k)} \cdot \beta_{kk}^{(k)} \;-\; \beta_{ik}^{(k)} \cdot \beta_{kj}^{(k)} \right) \;/\; \beta_{k-1,k-1}^{(k-1)}.
\end{aligned}$$

□

We reformulate Gaussian Elimination as shown earlier in Figure 2. This program first computes $\beta_{ij}^{(k+1)}$, and afterwards converts the results to the desired $a_{ij}^{(k+1)}$ values. The final loop, computing the u_{kj} values, is written to descend on k so that the u_{kj} and a_{kj} values can be co-located.

Theorem 5 *Assuming A is Gauss eliminable, standard Gaussian Elimination and the program in Figure 2 formally compute the same results.*

Proof is done by induction on k, requiring that we show:

$$\begin{aligned}
\ell_{ik} &= \beta_{ik}^{(k)} \;/\; \beta_{kk}^{(k)} \\
a_{ij}^{(k)} &= \beta_{ij}^{(k)} \;/\; \beta_{k-1,k-1}^{(k-1)}.
\end{aligned}$$

Both equations follow from the change of variables definition above. □

This reformulation again shows that there is an $LD^{-1}U$ decomposition for A that can be computed easily, this time from the $\beta_{ij}^{(k)}$ values (where neither L nor U is unit-triangular):

Theorem 6 *Any Gauss-eliminable matrix A has the decomposition*

$$A \;=\; \begin{pmatrix} \beta_{11}^{(1)} & & & & \\ \beta_{21}^{(1)} & \beta_{22}^{(2)} & & & \\ \beta_{31}^{(1)} & \beta_{32}^{(2)} & \beta_{33}^{(3)} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \beta_{n1}^{(1)} & \beta_{n2}^{(2)} & \beta_{n3}^{(3)} & \cdots & \beta_{nn}^{(n)} \end{pmatrix} \begin{pmatrix} 1/q_1 & & & & \\ & 1/q_2 & & & \\ & & 1/q_3 & & \\ & & & \ddots & \\ & & & & 1/q_n \end{pmatrix} \begin{pmatrix} \beta_{11}^{(1)} & \beta_{12}^{(1)} & \beta_{11}^{(1)} & \cdots & \beta_{1n}^{(1)} \\ & \beta_{22}^{(2)} & \beta_{23}^{(2)} & \cdots & \beta_{2n}^{(2)} \\ & & \beta_{33}^{(3)} & \cdots & \beta_{3n}^{(3)} \\ & & & \ddots & \vdots \\ & & & & \beta_{nn}^{(n)} \end{pmatrix}$$

where for $1 \leq k \leq n,\;\; q_k \;=\; \left(\beta_{kk}^{(k)}\; \beta_{k-1,k-1}^{(k-1)}\right)$.

Proof
As in Theorem 3 this follows from the decomposition $A = LU$ for standard Gaussian Elimination, because the equations $\ell_{ik} = \beta_{ik}^{(k)}/\beta_{kk}^{(k)}$, $a_{ij}^{(k)} = \beta_{ij}^{(k)}/\beta_{k-1,k-1}^{(k-1)}$ permit reorganizing $A = LU$ as $A = L_\beta \, D_\beta^{-1} \, U_\beta$ where

$$\begin{array}{lllllll} L_\beta & = & L \quad diag(\beta_{kk}^{(k)}) & = & (\beta_{ik}^{(k)}) & & \text{(where } i \geq k\text{: } L_\beta \text{ is lower triangular)} \\ D_\beta & = & \left(diag(\beta_{kk}^{(k)})^{-1} \;\; diag(\beta_{k-1,k-1}^{(k-1)})^{-1}\right)^{-1} & = & diag(\beta_{k-1,k-1}^{(k-1)} \; \beta_{kk}^{(k)}) & = & diag(q_k) \\ U_\beta & = & diag(\beta_{k-1,k-1}^{(k-1)}) \quad U & = & (\beta_{ij}^{(k)}) & & \text{(where } i \leq j\text{: } U_\beta \text{ is upper triangular).} \end{array}$$

□

Again the diagonals of L_β and U_β are identical, and *not* usually unit-valued. Also, when A is symmetric, we obtain $U_\beta = L_\beta{}^\top$.

4 What the Reformulations Compute

Theorems 3 and 6 can be derived directly from Sylvester's determinantal identity [9]. Basically, both reformulated algorithms implement what is called **pivotal condensation** [3].

Definition 2 *The* **submatrix of** A **with rows** $i_1, \ldots, i_p$ **and columns** $j_1, \ldots, j_q$ *(in that order) is denoted* $A[\, i_1, \ldots, i_p \,|\, j_1, \ldots, j_q \,]$*, and consists of the indicated elements of* A*:*

$$A[\, i_1, \ldots, i_p \,|\, j_1, \ldots, j_q \,] \quad = \quad \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_q} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_p,j_1} & a_{i_p,j_2} & \cdots & a_{i_p,j_q} \end{pmatrix}.$$

For example, the submatrix of A *with rows 3, 2 and columns 1, 3, 1, 4 is*

$$A[\, 3,2 \,|\, 1,3,1,4 \,] \quad = \quad \begin{pmatrix} a_{31} & a_{33} & a_{31} & a_{34} \\ a_{21} & a_{23} & a_{21} & a_{24} \end{pmatrix}.$$

Also, when $p = 0$ *or* $q = 0$ *the submatrix is 1 by convention, so* $A[\,|\,] = 1$.

Since it occurs often, we write

$$1..k \qquad \text{as synonymous for the sequence} \qquad 1,\, 2,\, \ldots,\, k.$$

Thus for example

$$A[\, 1..k, i \,|\, 1..k, j \,] \qquad \text{is synonymous for} \qquad A[\, 1, 2, \ldots, k, i \,|\, 1, 2, \ldots, k, j \,].$$

Theorem 7 *If* A *is an* $n \times n$ *Gauss eliminable matrix, then for* $1 \leq k < i \leq n$, $1 \leq k < j \leq n$*:*

$$\boxed{\beta_{ij}^{(k+1)} \quad = \quad det\ A[\, 1..k,\, i \,|\, 1..k,\, j \,]}$$

and if we define $p_0 = 1$ *and for* $1 \leq k < n$

$$p_k \quad = \quad \prod_{\kappa=1}^{k} \left(det\ A[\, 1..\kappa \,|\, 1..\kappa \,]\right)^{2^{\,k-\kappa}}$$

then

$$\boxed{\alpha_{ij}^{(k+1)} \quad = \quad p_{k-1} \;\cdot\; det\ A[\, 1..k,\, i \,|\, 1..k,\, j \,].}$$

A proof of this theorem is presented in the appendix.

5 Examples

Several familiar matrices give a better appreciation for the results above.

5.1 The Hilbert Matrix

The $n \times n$ Hilbert matrix $A = (a_{ij})$ is defined by

$$a_{ij} \quad = \quad \frac{1}{i+j-1}$$

It is a classical example of a poorly-conditioned matrix, with enormous condition number [10], although it is both positive definite and symmetric.

When $n = 4$, standard Gaussian Elimination obtains an LU factorization for A in 4 steps, as follows:

k	L	$U = A^{(k)}$
1	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 1/12 & 3/40 \\ 0 & 1/12 & 4/45 & 1/12 \\ 0 & 3/40 & 1/12 & 9/112 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 1 & 1 & 0 \\ 1/4 & 9/10 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 1/12 & 3/40 \\ 0 & 0 & 1/180 & 1/120 \\ 0 & 0 & 1/120 & 9/700 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 1 & 1 & 0 \\ 1/4 & 9/10 & 3/2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 1/12 & 3/40 \\ 0 & 0 & 1/180 & 1/120 \\ 0 & 0 & 0 & 1/2800 \end{pmatrix}.$

By contrast, the reformulated algorithms yield the following four iterations:

k	$\alpha^{(k)}$	d_k	$\beta^{(k)}$
1	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}$	1	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/12 & 1/12 & 3/40 \\ 1/3 & 1/12 & 4/45 & 1/12 \\ 1/4 & 3/40 & 1/12 & 9/112 \end{pmatrix}$	1/12	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/12 & 1/12 & 3/40 \\ 1/3 & 1/12 & 4/5 & 1/12 \\ 1/4 & 3/40 & 1/12 & 9/112 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/12 & 1/12 & 3/40 \\ 1/3 & 1/12 & 1/2160 & 1/1440 \\ 1/4 & 3/40 & 1/1440 & 3/2800 \end{pmatrix}$	1/25920	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/12 & 1/12 & 3/40 \\ 1/3 & 1/12 & 1/2160 & 1/1440 \\ 1/4 & 3/40 & 1/1440 & 3/2800 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/12 & 1/12 & 3/40 \\ 1/3 & 1/12 & 1/2160 & 1/1440 \\ 1/4 & 3/40 & 1/1440 & 1/72576000 \end{pmatrix}$	1/1881169920000	$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/12 & 1/12 & 3/40 \\ 1/3 & 1/12 & 1/2160 & 1/1440 \\ 1/4 & 3/40 & 1/1440 & 1/6048000 \end{pmatrix}$

Thus, for A we obtain the decomposition $L_\alpha D_\alpha^{-1} U_\alpha$ decomposition

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/12 & 0 & 0 \\ 1/3 & 1/12 & 1/2160 & 0 \\ 1/4 & 3/40 & 1/1440 & 1/72576000 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 25920 & 0 \\ 0 & 0 & 0 & 1881169920000 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 1/12 & 3/40 \\ 0 & 0 & 1/2160 & 1/1440 \\ 0 & 0 & 0 & 1/72576000 \end{pmatrix}$$

and the $L_\beta D_\beta^{-1} U_\beta$ decomposition

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/12 & 0 & 0 \\ 1/3 & 1/12 & 1/2160 & 0 \\ 1/4 & 3/40 & 1/1440 & 1/6048000 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 25920 & 0 \\ 0 & 0 & 0 & 13063680000 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 1/12 & 3/40 \\ 0 & 0 & 1/2160 & 1/1440 \\ 0 & 0 & 0 & 1/6048000 \end{pmatrix}.$$

Notice that, because A is symmetric, we in fact obtain an $LD^{-1}L^\top$ decomposition in both reformulations. Also notice that $\alpha_{34}^{(3)} = 1/1440$, $\alpha_{44}^{(4)} = 1/72576000$, $\beta_{34}^{(3)} = 1/1440$, $\beta_{44}^{(4)} = 1/6048000$ above, confirming Theorem 7, which asserts

$$\begin{aligned}
\alpha_{34}^{(3)} &= p_1 \cdot \mathit{det}\, A[1,2,3 \,|\, 1,2,4] = 1 \cdot 1/1440 = 1/1440 \\
\alpha_{44}^{(4)} &= p_2 \cdot \mathit{det}\, A[1,2,3,4 \,|\, 1,2,3,4] = 1/12 \cdot 1/6048000 = 1/72576000 \\
\beta_{34}^{(3)} &= \mathit{det}\, A[1,2,3 \,|\, 1,2,4] = 1/1440 \\
\beta_{44}^{(4)} &= \mathit{det}\, A[1,2,3,4 \,|\, 1,2,3,4] = 1/6048000.
\end{aligned}$$

5.2 The Wilkinson matrix

Consider the matrix presented by Wilkinson [12, p.212], adapted from a matrix of Turing [11], which gives worst-case error in LU decomposition with partial pivoting:

$$W_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 1 \\ -1 & -1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 \end{pmatrix}$$

When $n = 4$, the two reformulated algorithms obtain the same iterates for $A = W_4$:

k	$\alpha^{(k)}$	d_k	$\beta^{(k)}$
1	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 1 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix}$	1	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 1 & 2 \\ -1 & -1 & -1 & 2 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 1 & 4 \\ -1 & -1 & -1 & 4 \end{pmatrix}$	1	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 1 & 4 \\ -1 & -1 & -1 & 4 \end{pmatrix}$
4	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 1 & 4 \\ -1 & -1 & -1 & 8 \end{pmatrix}$	8	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ -1 & -1 & 1 & 4 \\ -1 & -1 & -1 & 8 \end{pmatrix}$

The $L_\alpha D_\alpha^{-1} U_\alpha$ and $L_\beta D_\beta^{-1} U_\beta$ decompositions are identical in this case:

$$W_4 \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

The iterates are identical since $p_k = 1$ for $0 \leq k < n$, so by Theorem 7, $\alpha_{ij}^{(k)} = \beta_{ij}^{(k)}$. Notice that here $\alpha_{34}^{(3)} = 4$, $\alpha_{44}^{(4)} = 8$, $\beta_{34}^{(3)} = 4$, $\beta_{44}^{(4)} = 8$, confirming Theorem 7, which asserts

$$\begin{array}{lcl} \alpha_{34}^{(3)} & = & p_1 \cdot det\ A[\,1,2,3\,|\,1,2,4\,] \quad = \quad 1 \cdot 4 \quad = \quad 4 \\ \alpha_{44}^{(4)} & = & p_2 \cdot det\ A[\,1,2,3,4\,|\,1,2,3,4\,] \quad = \quad 1 \cdot 8 \quad = \quad 8 \\ \beta_{34}^{(3)} & = & det\ A[\,1,2,3\,|\,1,2,4\,] \quad = \quad 4 \\ \beta_{44}^{(4)} & = & det\ A[\,1,2,3,4\,|\,1,2,3,4\,] \quad = \quad 8. \end{array}$$

6 Conclusion

We have presented two new reformulations of Gaussian Elimination with intriguing properties. One reformulation can give an $LD^{-1}U$ decomposition without using any division operations (ignoring inversion of D), while all intermediate results of the other are specific determinants of submatrices of the input matrix.

The practical usefulness of the reformulations in numerical computations remains to be determined. We have not yet studied their numerical properties. (It is worth mentioning that pivoting can be introduced into the reformulations, just as in standard Gaussian Elimination, because the $\alpha_{ij}^{(k)}$, $\beta_{ij}^{(k)}$, and $a_{ij}^{(k)}$ values are closely related, and we always seek nonzero diagonal elements.) This paper is to report on the existence of the reformulations, some of their interesting aspects, and their natural characterization in terms of determinantal identities.

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Appendix: Determinantal Formulas for Gaussian Elimination

To prove Theorem 7, we need several 19th-century identities concerning determinants. We take the time to prove them carefully because they are instructive, and they clarify the relationship between Gaussian Elimination and determinants. Also, the proofs dispel the magic that otherwise surrounds these identities.

Theorem 8 *(Chiò, 1853 [1])*
If A is an $n \times n$ Gauss eliminable matrix, then

$$det\ A \;=\; \frac{1}{a_{11}^{n-2}}\ det \begin{pmatrix} det\ A[\,1,2\,|\,1,2\,] & det\ A[\,1,2\,|\,1,3\,] & \cdots & det\ A[\,1,2\,|\,1,n\,] \\ det\ A[\,1,3\,|\,1,2\,] & det\ A[\,1,3\,|\,1,3\,] & \cdots & det\ A[\,1,3\,|\,1,n\,] \\ \vdots & \vdots & \ddots & \vdots \\ det\ A[\,1,n\,|\,1,2\,] & det\ A[\,1,n\,|\,1,3\,] & \cdots & det\ A[\,1,n\,|\,1,n\,] \end{pmatrix}.$$

Proof
When A is an $n \times n$ Gauss eliminable matrix, $a_{11} \neq 0$. The theorem can then be proven with invariance of the determinant under elementary row operations. Multiplying all but the first row of A by a_{11} augments the determinant by a_{11}^{n-1}. Then Gaussian Elimination on the first column leaves this determinant value undisturbed.

$$det\ A \;=\; \frac{1}{a_{11}^{n-1}}\ det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} & \cdots & a_{11}a_{2n} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} & \cdots & a_{11}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{11}a_{n1} & a_{11}a_{n2} & a_{11}a_{n3} & \cdots & a_{11}a_{nn} \end{pmatrix}$$

$$= \frac{1}{a_{11}^{n-1}}\ det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & (a_{11}a_{22}-a_{21}a_{12}) & (a_{11}a_{23}-a_{21}a_{13}) & \cdots & (a_{11}a_{2n}-a_{21}a_{1n}) \\ 0 & (a_{11}a_{32}-a_{31}a_{12}) & (a_{11}a_{33}-a_{31}a_{13}) & \cdots & (a_{11}a_{3n}-a_{31}a_{1n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (a_{11}a_{n2}-a_{n1}a_{12}) & (a_{11}a_{n3}-a_{n1}a_{13}) & \cdots & (a_{11}a_{nn}-a_{n1}a_{1n}) \end{pmatrix}$$

$$= \frac{1}{a_{11}^{n-2}}\ det \begin{pmatrix} det\ A[\,1,2\,|\,1,2\,] & det\ A[\,1,2\,|\,1,3\,] & \cdots & det\ A[\,1,2\,|\,1,n\,] \\ det\ A[\,1,3\,|\,1,2\,] & det\ A[\,1,3\,|\,1,3\,] & \cdots & det\ A[\,1,3\,|\,1,n\,] \\ \vdots & \vdots & \ddots & \vdots \\ det\ A[\,1,n\,|\,1,2\,] & det\ A[\,1,n\,|\,1,3\,] & \cdots & det\ A[\,1,n\,|\,1,n\,] \end{pmatrix}.$$

□

Chiò's pivotal condensation process uses this identity iteratively, repeatedly reducing the evaluation of an $n \times n$ determinant to an $(n-1) \times (n-1)$ determinant. The reformulations of Gaussian Elimination in this paper manifest what Householder [6, p.141] called "the essential identity of Gaussian Elimination and the method of Chió for evaluating a determinant".

When taken to the limit, Chiò's process yields Sylvester's identity:

Theorem 9 *(Sylvester, 1851 [9])*
Let A be an $n \times n$ Gauss eliminable matrix. Then for $1 \leq k < i \leq n$, $1 \leq k < j \leq n$:

$$\begin{aligned} & det\ A[\,1..(k-1)\,|\,1..(k-1)\,] \ \cdot\ det\ A[\,1..(k-1),k,i\,|\,1..(k-1),k,j\,] \\ &= \ det \begin{pmatrix} det\ A[\,1..(k-1),k\,|\,1..(k-1),k\,] & det\ A[\,1..(k-1),k\,|\,1..(k-1),j\,] \\ det\ A[\,1..(k-1),i\,|\,1..(k-1),k\,] & det\ A[\,1..(k-1),i\,|\,1..(k-1),j\,] \end{pmatrix}. \end{aligned}$$

Proof
We need only prove the memorable identity obtained with $k = (n-1)$:

$$\begin{aligned} & det\ A[\,1..(n-2)\,|\,1..(n-2)\,] \ \cdot \ det\ A[\,1..n\,|\,1..n\,] \\ = \ & det \begin{pmatrix} det\ A[\,1..(n-1)\,|\,1..(n-1)\,] & det\ A[\,1..(n-1)\,|\,1..(n-2),n\,] \\ det\ A[\,1..(n-2),n\,|\,1..(n-1)\,] & det\ A[\,1..(n-2),n\,|\,1..(n-2),n\,] \end{pmatrix}. \end{aligned}$$

This identity is equivalent to the statement of the theorem, since when $k < (n-1)$ we can replace A with the $(k+1)\times(k+1)$ submatrix $A[\,1..(k-1),k,i\,|\,1..(k-1),k,j\,]$, and view n as $(k+1)$. (With this in mind, Sylvester stated the theorem in a more general form, using expressions such as $det\ A[\,a_1,\ldots,a_n\,|\,\alpha_1,\ldots,\alpha_n\,]$ where $a_1,\ldots,a_n,\alpha_1,\ldots,\alpha_n$ represent arbitrary indices.)

The theorem can be proved by a careful induction on n. It uses the statement of the theorem as an induction hypothesis, but establishes only the memorable identity above, relying on their equivalence to complete the proof.

For the basis $n = 2$ (forcing $k = 1$), the theorem asserts that

$$det\ A[\ |\ ]\cdot det\ A[\,1,2\,|\,1,2\,] \quad = \quad det\ A[\,1\,|\,1\,]\cdot det\ A[\,2\,|\,2\,] \ - \ det\ A[\,2\,|\,1\,]\cdot det\ A[\,1\,|\,2\,]$$

i.e., simply that $1\cdot(a_{11}a_{22} - a_{21}a_{12}) = a_{11}\cdot a_{22} - a_{21}\cdot a_{12}$.

For the induction step, the induction hypothesis (the statement of the theorem for $k < (n-1)$) is sufficient to prove the memorable identity. In fact, the final case is the final result of Chiò's process, and this is reflected in its proof. We claim that for $2 \le k \le n-1$:

$$\begin{aligned} det\ A \ = \ & 1/(det\ A[\,1..(k-1)\,|\,1..(k-1)\,])^{n-k} \\ \cdot\ det & \begin{pmatrix} det\ A[\,1..(k-1),\ (k)\ \,|\,1..(k-1),k\,] & \cdots & det\ A[\,1..(k-1),\ (k)\ \,|\,1..(k-1),n\,] \\ det\ A[\,1..(k-1),(k+1)\,|\,1..(k-1),k\,] & \cdots & det\ A[\,1..(k-1),(k+1)\,|\,1..(k-1),n\,] \\ \vdots & \ddots & \vdots \\ det\ A[\,1..(k-1),\ (n)\ \,|\,1..(k-1),k\,] & \cdots & det\ A[\,1..(k-1),\ (n)\ \,|\,1..(k-1),n\,] \end{pmatrix}. \end{aligned}$$

The case $k = 2$ is Theorem 8, and the case $k = (n-1)$ is the memorable identity above, so proving the claim will establish the theorem.

The claim can be proven with a nested induction on k, using Chiò's basis $k = 2$. The induction hypothesis allows us to implement Chiò's process, increasing k from 2 to $n-1$. Each step reduces the right side of the claim, $1/(det\ A[\,1..(k-1)\,|\,1..(k-1)\,])^{n-k}$ multiplied by the determinant of an $(n-k+1)\times(n-k+1)$ submatrix, to $1/(det\ A[\,1..k\,|\,1..k\,])^{n-(k+1)}$ multiplied by the determinant of an $(n-k)\times(n-k)$ submatrix. The [1,1] entry of the former submatrix is $det\ A[\,1..k\,|\,1..k\,]$. Following the proof of Theorem 8, the determinant of this submatrix is $1/(det\ A[\,1..k\,|\,1..k\,])^{n-(k+1)}$ multiplied by the determinant of an $(n-k)\times(n-k)$ matrix, in which each entry is a 2×2 determinant. The induction hypothesis can then be used to replace the $(i-k+1),(j-k+1)$-th 2×2 determinant (because its entries are determinants of $k\times k$ submatrices with $k < (n-1)$) by the product of two terms, $det\ A[\,1..(k-1)\,|\,1..(k-1)\,]$ and $det\ A[\,1..(k-1),k,i\,|\,1..(k-1),k,j\,]$. Factoring out the terms $det\ A[\,1..(k-1)\,|\,1..(k-1)\,]$ cancels the previous leading multiplicative factor $1/(det\ A[\,1..(k-1)\,|\,1..(k-1)\,])^{n-k}$ and yields the claim for $k+1$. □

This brings us to the theorem we set out to prove: all intermediate results of the reformulations of Gaussian Elimination are expressible as (products of) determinants of submatrices of A.

Theorem 7 *If A is an $n\times n$ Gauss eliminable matrix, then for $1 \le k < i \le n$, $1 \le k < j \le n$:*

$$\beta_{ij}^{(k+1)} \quad = \quad det\ A[\,1..k,\ i\,|\,1..k,\ j\,].$$

$$\alpha_{ij}^{(k+1)} \quad = \quad p_{k-1} \ \cdot \ det\ A[\,1..k,\ i\,|\,1..k,\ j\,]$$

where for $0 \le k \le n$

$$p_k \quad = \quad \prod_{\kappa=1}^{k} \; (det\ A[\,1..\kappa \,|\, 1..\kappa\,])^{2^{\,k-\kappa}}.$$

Proof
By induction on n. For $n = 2$, necessarily $k = 1$, $i = j = 2$, and we require only the verification

$$\beta_{22}^{(2)} \quad = \quad \left(\beta_{11}^{(1)} \cdot \beta_{22}^{(1)} \;-\; \beta_{21}^{(1)} \cdot \beta_{12}^{(1)}\right) \;/\; \beta_{00}^{(0)} \quad = \quad a_{11} \cdot a_{22} - a_{21} \cdot a_{12} \quad = \quad det\ A[\,1,2 \,|\, 1,2\,].$$

The induction step also holds for β since

$$\begin{aligned}
\beta_{ij}^{(k+1)} \quad &= \quad \left(\beta_{ij}^{(k)} \cdot \beta_{kk}^{(k)} \;-\; \beta_{ik}^{(k)} \cdot \beta_{kj}^{(k)}\right) \;/\; \beta_{(k-1),(k-1)}^{(k-1)} \\
&= \quad (det\ A[\,1..(k-1), i \,|\, 1..(k-1), j\,] \cdot det\ A[\,1..(k-1) \,|\, 1..(k-1)\,] \\
&\qquad - det\ A[\,1..(k-1), i \,|\, 1..(k-1), k\,] \cdot det\ A[\,1..(k-1), k \,|\, 1..(k-1), j\,]) \\
&\qquad\quad /\ det\ A[\,1..(k-1) \,|\, 1..(k-1)\,] \\
&= \quad (det\ A[\,1..(k-1) \,|\, 1..(k-1)\,] \cdot det\ A[\,1..k, i \,|\, 1..k, j\,]) \qquad \text{(by Theorem 9)} \\
&\qquad\quad /\ det\ A[\,1..(k-1) \,|\, 1..(k-1)\,] \\
&= \quad det\ A[\,1..k, i \,|\, 1..k, j\,].
\end{aligned}$$

The second statement also follows by induction on n. When $n = 2$, again $k = 1$, $i = j = 2$, and we need only the verification

$$\alpha_{22}^{(2)} \quad = \quad \alpha_{11}^{(1)} \cdot \alpha_{22}^{(1)} \;-\; \alpha_{21}^{(1)} \cdot \alpha_{12}^{(1)} \quad = \quad a_{11} \cdot a_{22} - a_{21} \cdot a_{12} \quad = \quad p_0 \cdot det\ A[\,1,2 \,|\, 1,2\,].$$

For the induction step, notice that $p_k = p_{k-1}^2 \cdot det\ A[\,1..k \,|\, 1..k\,]$, so

$$\begin{aligned}
\alpha_{ij}^{(k+1)} \quad &= \quad \alpha_{ij}^{(k)} \cdot \alpha_{kk}^{(k)} \;-\; \alpha_{ik}^{(k)} \cdot \alpha_{kj}^{(k)} \\
&= \quad p_{k-2}^2 \cdot (det\ A[\,1..(k-1), i \,|\, 1..(k-1), j\,] \cdot det\ A[\,1..(k-1) \,|\, 1..(k-1)\,] \\
&\qquad\qquad - det\ A[\,1..(k-1), i \,|\, 1..(k-1), k\,] \cdot det\ A[\,1..(k-1), k \,|\, 1..(k-1), j\,]) \\
&= \quad p_{k-2}^2 \cdot (det\ A[\,1..(k-1) \,|\, 1..(k-1)\,] \cdot det\ A[\,1..k, i \,|\, 1..k, j\,]) \qquad \text{(by Theorem 9)} \\
&= \quad p_{k-1} \cdot det\ A[\,1..k, i \,|\, 1..k, j\,].
\end{aligned}$$

□

For more perspective on determinantal identities and Gaussian Elimination, see [7].

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