Schur Complements obey Lambek's Categorial Grammar: another view of Gaussian Elimination and LU Decomposition

D. Stott Parker
stott@cs.ucla.edu
Computer Science Department
University of California
Los Angeles, CA 90024-1596

October 20, 1995

Abstract

For two decades Schur complements have seen increasing applications in linear algebra, often as abstractions of Gaussian elimination. It is known that they obey certain nontrivial identities, such as Crabtree and Haynsworth's *quotient property*. We began this work asking if there were a theory for deciding their properties in general.

Lambek's Categorial Grammar is a deductive system introduced in 1958 by Lambek as a mathematical foundation for a syntactic calculus of English sentences. We show that Categorial Grammar gives a deductive system for deciding properties LU- and UL-decompositions, and about Gaussian elimination. It also can be used to derive identities obeyed by Schur complements.

At first impression this seems to be a strange result, connecting two unrelated topics. In retrospect, though, it is a consequence of the way both use quotients. It may have applications in developing both grammatical formalisms and numerical algorithms.

Keywords: Schur complements, Quotients, Categorial Grammar, Categorical Grammar, Lambek Calculus, Gaussian elimination, LU decomposition.

1 Introduction

Although Schur complements are mentioned in almost every modern text on computational linear algebra, their literature is less than thirty years old and is still very incomplete. For example, Crabtree and Haynsworth [4] first noted the 'quotient identity' for Schur complements (discussed below) in 1969, and there have been many papers relating them to generalized inverses, but beyond Cottle's 1974 survey [3] we have found no ensuing general work characterizing the identities obeyed by Schur complements. The subject of Schur complements is important in numerical analysis, and asks for deeper investigation. It is a rich subject, with connections to the theory of quotients over modules and homological algebra.

Lambek introduced Categorial Grammar in 1958 [11] as a syntactic calculus for English sentences, an outcome of his "observation that a notation which was useful in two branches of algebra, module theory and ideal theory ..., could also be applied to the study of sentence structure in natural languages" [13]. It is a very simple yet powerful deductive system that has inspired new directions in computational linguistics. An excellent summary of recent work on Lambek calculi and related systems is the edited volume [16].

Recently we derived a number of new results about Schur complements and Gaussian elimination [19, 20, 21, 22]. This paper just demonstrates that matrix decompositions using Schur complements (and thus the decompositions obtained with Gaussian elimination) obey exactly the rules of Categorial Grammar, giving a new tool for reasoning about these decompositions.

2 Schur Complements

2.1 Terminology

Definition 1 *The* **submatrix of** A **with rows** $i_1, \ldots, i_p$ **and columns** $j_1, \ldots, j_q$ *(in that order) is denoted* $A[\, i_1, \ldots, i_p \,|\, j_1, \ldots, j_q \,]$*, and consists of the indicated elements of* A*:*

$$A[\, i_1, \ldots, i_p \,|\, j_1, \ldots, j_q \,] \quad = \quad \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_q} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_p,j_1} & a_{i_p,j_2} & \cdots & a_{i_p,j_q} \end{pmatrix}.$$

For example, the submatrix of A *with rows 3, 2 and columns 1, 3, 1, 4 is*

$$A[\, 3,2 \,|\, 1,3,1,4 \,] \quad = \quad \begin{pmatrix} a_{31} & a_{33} & a_{31} & a_{34} \\ a_{21} & a_{23} & a_{21} & a_{24} \end{pmatrix}.$$

Also, when $p = 0$ *or* $q = 0$ *the submatrix is 1 by convention, so* $A[\;|\;] = 1$*. Generally we will refer to the empty matrix (the* 0×0 *matrix) as '1'.*

Definition 2 $X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ *is the* **matrix direct sum** *of square matrices* X *and* Y*. Thus* $1 \oplus X = X \oplus 1 = X$*.*

Definition 3 *An* $n \times n$ *matrix* A *is* **nondegenerate** *if det* $A[\, \alpha \,|\, \alpha \,] \neq 0$*, for every subsequence* α *of* $1, \ldots, n$*.*

Throughout this paper we will require all matrices to be nondegenerate. This requirement is not as restrictive as it may seem, since any nonsingular linear system $A\mathbf{x} = \mathbf{b}$ can be transformed to an equivalent nondegenerate system $\widetilde{A}\mathbf{x} = \widetilde{\mathbf{b}}$ via appropriate random matrices [19, 21].

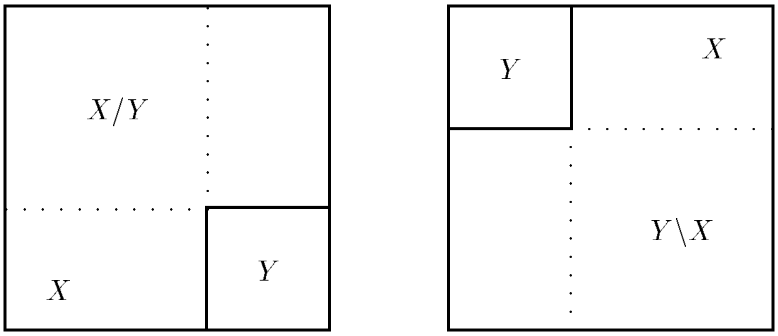
Definition 4 *if* X *and* Y *are square matrices, define*

$$\begin{array}{lll} Y \trianglelefteq X & \textit{iff} & Y \textit{ is a nonsingular leading principal submatrix of } X. \\ X \trianglerighteq Y & \textit{iff} & Y \textit{ is a nonsingular trailing principal submatrix of } X. \end{array}$$

For example, if Y is nonsingular, $Y \trianglelefteq (Y \oplus X)$ and $(X \oplus Y) \trianglerighteq Y$.

2.2 The Schur Complement and its Properties

Emilie Haynsworth pioneered interest in the Schur complement in a series of papers published between 1968 and 1974 [2, 4, 6, 7, 8, 17]. Cottle [3] surveys various places where Schur complements turn up, including determinantal identities, pivoting, inverse matrices, inertia, etc. See also [9, pp.17–23]. Here we use a definition in which the ordering of rows and columns is significant.



$X \trianglerighteq Y$ $\qquad\qquad$ $Y \trianglelefteq X$

Figure 1: The Schur complement of a trailing or leading principal submatrix Y in X

Definition 5 *If a nondegenerate matrix X has block decomposition $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, so that $A \trianglelefteq X$ and $X \trianglerighteq D$ then:*
the **trailing Schur complement of D in X**, *written $(\ X \ / \ D\ )$, is*

$$(\ X \ / \ D\ ) \quad = \quad A - BD^{-1}C,$$

and the **leading Schur complement of A in X**, *written $(\ A \setminus X\ )$, is*

$$(\ A \setminus X\ ) \quad = \quad D - CA^{-1}B.$$

Schur complements can be understood pictorially. In Figure 1, the dotted outlines in the diagrams delimit the complement of the submatrix Y in X, if for example the off-diagonal blocks are zero. (Non-visually-oriented readers can ignore these figures without really losing anything, but they are handy on occasion, especially in reasoning about block matrix decompositions later.)

Lemma 1 (Schur's identity for determinants [23])
If $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, so that $A \trianglelefteq X$ and $X \trianglerighteq D$, then

$$\begin{aligned} det\ X &= det\ (\ X \ / \ D\ ) \quad det\ D \\ det\ X &= det\ A \quad det\ (\ A \setminus X\ ). \end{aligned}$$

These identities are derivable easily with a block LU decomposition of X.

Lemma 2 (Quotient Property of Schur Complements)
If X is a nondegenerate matrix such that

$$X \quad = \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A \quad = \quad \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad D \quad = \quad \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

are block decompositions with $A_{11} \trianglelefteq A \trianglelefteq X$ and $X \trianglerighteq D \trianglerighteq D_{22}$, then

$$\begin{aligned} (\ X \ / \ D\ ) &= (\ (\ X \ / \ D_{22}\ ) \ / \ (\ D \ / \ D_{22}\ )\ ) \\ (\ A \setminus X\ ) &= (\ (\ A_{11} \setminus A\ ) \setminus (\ A_{11} \setminus X\ )\ ). \end{aligned}$$

This was proven in [4], using Schur's identity for determinants (Lemma 1). The proof is complicated. Another proof by Ostrowski [17] is also complicated.

2.3 Gaussian Elimination and Schur Complements

In its basic form, Gaussian elimination is a sequence of transformations to an $n \times n$ square matrix $A = (a_{ij})$, reducing it to upper-diagonal form in n steps. It can be defined equationally, with the initial assignment $a_{ij}^{(1)} = a_{ij}$ and the recursive definition of $a_{ij}^{(k+1)}$ for $1 \leq k \leq n-1$:

$$a_{ij}^{(k+1)} \quad = \quad \begin{cases} 0 & i \geq k+1,\ j = k \\ a_{ij}^{(k)} \; - \; a_{ik}^{(k)} \; \left(a_{kk}^{(k)}\right)^{-1} \; a_{kj}^{(k)} & i \geq k+1,\ j \geq k+1 \\ a_{ij}^{(k)} & \text{otherwise.} \end{cases}$$

Often this definition is viewed as summarizing a program – an implementation of Gaussian elimination. However: in this paper, the definition is viewed as a set of equations, without any 'round-off error'. Gaussian elimination is typically implemented with **for**–loops in a program requiring $n(n^2-1)/3$ assignments altogether:

for $k = 1$ **to** $n-1$ **do**
  for $i = k+1$ **to** n **do**
    begin
      $m_{ik} \; := \; a_{ik} \; \div \; a_{kk};$ $\{ \quad a_{ik}^{(k)} \; \div \; a_{kk}^{(k)} \quad \}$
      for $j = k+1$ **to** n **do**
        $a_{ij} \; := \; a_{ij} \; - \; m_{ik} \; \times \; a_{kj}$ $\{ \quad a_{ij}^{(k)} \; - \; a_{ik}^{(k)} \; \div \; a_{kk}^{(k)} \; \times \; a_{kj}^{(k)} \quad \}$
    end

Starting with $A^{(1)} = A$, each iteration of the outer loop computes $A^{(k+1)} = (a_{ij}^{(k+1)})$ for $1 \leq k \leq n-1$, except that as written above the program does not zero the elements of A below the diagonal in the first k columns. (These elements are in precisely the same positions as the multipliers m_{ik}, so many implementations store m_{ik} in a_{ik}.) With the program we can form the triangular matrices

$$L \quad = \quad \begin{pmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ m_{n1} & m_{n2} & m_{n3} & \cdots & 1 \end{pmatrix}, \qquad U \quad = \quad \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} & a_{13}^{(n)} & \cdots & a_{1n}^{(n)} \\ & a_{22}^{(n)} & a_{23}^{(n)} & \cdots & a_{2n}^{(n)} \\ & & a_{33}^{(n)} & \cdots & a_{3n}^{(n)} \\ & & & \ddots & \vdots \\ & & & & a_{nn}^{(n)} \end{pmatrix}$$

such that $A^{(n)} = U$ and $A = LU$. Gaussian elimination works only when the $a_{kk}^{(k)}$ are nonzero, for $1 \leq k \leq n$, which holds when A is nondegenerate [20]. In this paper we will not consider pivoting or other extensions of Gaussian elimination for degenerate matrices.

Definition 6 *Let* $A_{(1)} = A$ *and, for* $1 \leq k < n$,

$$A_{(k+1)} \quad \stackrel{def}{=} \quad A^{(k+1)}\,[\,(k+1),\ldots,n\,|\,(k+1),\ldots,n\,],$$

i.e., $A_{(k+1)}$ *is the* $(n-k)\times(n-k)$ *submatrix* $(a_{ij}^{(k+1)})$ *where* $1 \leq k < i, j \leq n$.

Theorem 1 *If* A *is an* $n \times n$ *nondegenerate matrix, then* $A_{(k+1)} \; = \; (\; A[\,1,\ldots,k\,|\,1,\ldots,k\,] \setminus A\;)$. *Equivalently:* $\; a_{ij}^{(k+1)} \;\; = \;\; (\;\; A[\,1,\ldots,k\,|\,1,\ldots,k\,] \;\; \setminus \;\; A[\,1,\ldots,k,i\,|\,1,\ldots,k,j\,] \;\;)$.

Proved in [20]. Thus every result of Gaussian elimination is expressible as a Schur complement.

Corollary 1 *If A is an $n \times n$ nondegenerate matrix, then* $A_{(k+1)} = (\ (\ a_{kk}^{(k)}\ ) \setminus A_{(k)}\ ),\ \ 1 \leq k < n.$

This identity just combines Lemma 2 (the quotient property) and Theorem 1. It gives an incremental algorithm for computing $A_{(k+1)}$:

$$
\begin{array}{l}
A_{(1)} := A\ ; \\
\textbf{for } k = 1 \textbf{ to } n-1 \textbf{ do} \\
\quad \textbf{begin} \\
\qquad a_{kk}^{(k)} := A_{(k)}[\,1\,|\,1\,]\ ; \\
\qquad A_{(k+1)} := (\ (\ a_{kk}^{(k)}\ ) \setminus A_{(k)}\ ) \\
\quad \textbf{end}\ .
\end{array}
$$

This is exactly Gaussian elimination.

3 Lambek's Categorial Grammar

In Categorial Grammar, English words are assigned "**categories**", which are intended as "types", in the philosophical sense attributed to Aristotle and Kant. We can introduce a **lexicon** — a type assignment between English words and types — such as the following:

Word	*Type*
equations	N
Gauss	NP
linear	N/N
quickly	S\S
solved	NP\(S/NP)
slept	NP\S
the	NP/N

Each type is a functional *pattern*, describing the role or effect of the word in a sentence. The word "Gauss" is a Noun Phrase. Verbs such as "slept" yield sentences when composed with noun phrases, so intuitively they have a type like "NP $\mapsto$ S". In Categorial Grammar this type is either NP\S or S/NP, which also captures insistence on word orderings. Specifically, NP\S maps an *immediately preceding* NP to an S, while S/NP would map an *immediately following* NP to an S, so the word "slept" of type NP\S yields a sentence if there is a preceding noun phrase, but not when there is a following noun phrase.

The preceding/following ordering relation is the precedence relation among elements in sequences, and here sequences are constructed using the associative '·' operator. As an example, (S/NP)·NP·(S\S) is such a sequence, and in it S/NP has a following NP, and NP has a following S\S.

Generally a **type** is defined to be either an **atomic type** (a symbol like NP or S) or a **compound type** (an expression made up of types and the binary operators '·', '\', and '/'). English language categories (verb, adjective, etc.) are typically identifiable with a small number of types, and a particular word can therefore have more than one type. For example, adverbs like "quickly" could be identified with either S\S or S/S.

Types obey certain laws. For example we would expect adverbs, of type S\S, also to be of type (NP\S)\(NP\S), mapping verbs to verbs. Categorial Grammar defines reduction laws with which these intuitive subsumptions among types can be derived formally, and with which types can be **reduced** (simplified, or parsed). In [13], Lambek gives axioms and rules of inference defining the relation '$\Rightarrow$' on types X, Y, Z where 1 is a special identity type:

Axioms	$(X \cdot 1) \Rightarrow X$ $X \Rightarrow (X \cdot 1)$ $(1 \cdot X) \Rightarrow X$ $X \Rightarrow (1 \cdot X)$ $((X \cdot Y) \cdot Z) \Rightarrow (X \cdot (Y \cdot Z))$ $(X \cdot (Y \cdot Z)) \Rightarrow ((X \cdot Y) \cdot Z)$		
Rules of Inference	if $X \Rightarrow Y$ and $Y \Rightarrow Z$	then	$X \Rightarrow Z$
	if $(Y \cdot Z) \Rightarrow X$	then	$Y \Rightarrow (X/Z)$
	if $Y \Rightarrow (X/Z)$	then	$(Y \cdot Z) \Rightarrow X$
	if $(Y \cdot Z) \Rightarrow X$	then	$Z \Rightarrow (Y \backslash X)$
	if $Z \Rightarrow (Y \backslash X)$	then	$(Y \cdot Z) \Rightarrow X$

From these rules '$\Rightarrow$' can be seen to be reflexive (the first two axioms and first inference rule give $X \Rightarrow X$), and transitive (first inference rule). Categorial Grammar is sometimes defined differently, but this definition is suitable for our needs.

With this formal system we can derive many interesting laws. For example, using A and B as type variables instead of X and Y to avoid confusion, we can derive the so-called 'type raising' law

$$A \quad \Rightarrow \quad ((B \,/\, A) \,\backslash\, B)$$

in the following way:

1.	$B/A \;\Rightarrow\; ((B/A) \cdot 1)$	axiom: $X \Rightarrow (X \cdot 1)$
2.	$((B/A) \cdot 1) \;\Rightarrow\; B/A$	axiom: $(X \cdot 1) \Rightarrow X$
3.	$B/A \;\Rightarrow\; B/A$	1. & 2. & rule: if $X \Rightarrow Y$ and $Y \Rightarrow Z$ then $X \Rightarrow Z$
4.	$((B/A) \cdot A) \;\Rightarrow\; B$	3. & rule: if $Y \Rightarrow (X/Z)$ then $(Y \cdot Z) \Rightarrow X$
5.	$A \;\Rightarrow\; ((B/A)\backslash B)$	4. & rule: if $(Y \cdot Z) \Rightarrow X$ then $Z \Rightarrow (Y \backslash X)$.

In the same way we can derive each of the following reduction laws [14, p.11ff]:

Application	$(X \,/\, Y) \cdot Y$	$\Rightarrow$	X
	$Y \cdot (Y \,\backslash\, X)$	$\Rightarrow$	X
Composition	$(X \,/\, Y) \cdot (Y \,/\, Z)$	$\Rightarrow$	$(X \,/\, Z)$
	$(Z \,\backslash\, Y) \cdot (Y \,\backslash\, X)$	$\Rightarrow$	$(Z \,\backslash\, X)$
Associativity	$(Z \,\backslash\, X) \,/\, Y$	$\Rightarrow$	$Z \,\backslash\, (X \,/\, Y)$
	$Z \,\backslash\, (X \,/\, Y)$	$\Rightarrow$	$(Z \,\backslash\, X) \,/\, Y$
Type Raising	X	$\Rightarrow$	$Y \,/\, (X \,\backslash\, Y)$
	X	$\Rightarrow$	$(Y \,/\, X) \,\backslash\, Y$
Quotient	$(X \,/\, Y)$	$\Rightarrow$	$(X \,/\, Z) \,/\, (Y \,/\, Z)$
	$(Y \,/\, Z)$	$\Rightarrow$	$(X \,/\, Y) \,\backslash\, (X \,/\, Z)$
	$(Y \,\backslash\, X)$	$\Rightarrow$	$(Z \,\backslash\, Y) \,\backslash\, (Z \,\backslash\, X)$
	$(Z \,\backslash\, Y)$	$\Rightarrow$	$(Z \,\backslash\, X) \,/\, (Y \,\backslash\, X)$.

For example, the four Quotient laws are derivable directly from the Composition laws with the rules of inference.

In addition, we can derive rules of inference, including [13, p.303]:

$$
\begin{array}{llll}
\text{if} & Y \Rightarrow Z & \text{then} & (X \cdot Y) \Rightarrow (X \cdot Z) \\
\text{if} & Y \Rightarrow Z & \text{then} & (Y \cdot X) \Rightarrow (Z \cdot X) \\[4pt]
\text{if} & Y \Rightarrow Z & \text{then} & (X \,/\, Z) \Rightarrow (X \,/\, Y) \\
\text{if} & Y \Rightarrow Z & \text{then} & (Y \,/\, X) \Rightarrow (Z \,/\, X) \\[4pt]
\text{if} & Y \Rightarrow Z & \text{then} & (Z \,\backslash\, X) \Rightarrow (Y \,\backslash\, X) \\
\text{if} & Y \Rightarrow Z & \text{then} & (X \,\backslash\, Y) \Rightarrow (X \,\backslash\, Z) \;.
\end{array}
$$

It is convenient to use these derived laws and rules of inference in order to parse a sequence of words by reducing it to the type 'sentence' (represented above by 'S'). The following example shows how this can work, repeatedly using the derived rules of inference, the indicated derived laws (on the underlined subexpression), and transitivity:

	Gauss	·	solved	·	the	·	linear	·	equations	·	quickly
	NP	·	NP\(S/NP)	·	NP/N	·	N/N	·	N	·	S\S
⇒Composition	NP	·	NP\(S/NP)	·	NP/N			·	N	·	S\S
⇒Application	NP	·	NP\(S/NP)	·			NP			·	S\S
⇒Associativity	NP	·	(NP\S)/NP	·			NP			·	S\S
⇒Application	NP	·			(NP\S)					·	S\S
⇒Quotient	NP	·			(NP\S)					·	(NP\S)\(NP\S)
⇒Application	NP	·							NP\S		
⇒Application					S						

In other words, this derivation gives the sketch of a proof that

$$\mathsf{NP} \cdot \mathsf{NP}\backslash(\mathsf{S}/\mathsf{NP}) \cdot \mathsf{NP}/\mathsf{N} \cdot \mathsf{N}/\mathsf{N} \cdot \mathsf{N} \cdot \mathsf{S}\backslash\mathsf{S} \quad \Rightarrow \quad \mathsf{S}.$$

A **categorial grammar** for a language is the syntactic calculus above, along with a lexicon that assigns types to individual words in the language. With this grammar we can introduce a suitable definition for sentences in terms of types. For example, usually sequences of words $(w_1 \cdot w_2 \cdots w_n)$ whose lexical types $X_1, \ldots, X_n$ satisfy $(X_1 \cdot X_2 \cdots X_n) \Rightarrow \mathsf{S}$ are called **sentences** of the language.

By now the reader will have gotten the drift here — clearly there is a strong parallel between Schur complements and Lambek's Categorial Grammar. The only question is how to formalize the connection.

4 zULU Decompositions and zULU Classes

Our idea is to view an individual matrix as generating a set of its 'block unit diagonalizations', which are block diagonal matrix decompositions using Schur complements. These sets have a natural quotient structure like that behind Categorial Grammar.

Definition 7 *An $n \times n$* **block unit lower triangular matrix** *L has form $\begin{pmatrix} I_m & 0 \\ M & I_{m-n} \end{pmatrix}$ for some $1 \leq m < n$, where I_m is an $m \times m$ identity matrix, and M is an $(n-m) \times m$ arbitrary matrix. Similarly, a* **block unit upper triangular matrix** *U has form $\begin{pmatrix} I_{m-n} & M \\ 0 & I_m \end{pmatrix}$.*

The block matrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has the **block unit UL diagonalization** *$(\ X \ / \ D\ ) \oplus D$, and the* **block unit LU diagonalization** *$A \oplus (\ A \setminus X\ )$, provided A and D are square and nonsingular.*

These definitions reflect the following decompositions:

$$\begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} = \begin{pmatrix} (A - BD^{-1}C) & 0 \\ 0 & D \end{pmatrix} = (\ X \ / \ D\ ) \oplus D,$$

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (D - CA^{-1}B) \end{pmatrix} = A \oplus (\ A \setminus X\ ).$$

For example, the matrix

$$X \;=\; \begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 7 \\ 1 & 2 & 3 \end{pmatrix}$$

with determinant 1 has the following block unit diagonalizations:

$$\begin{array}{llll}
(\, X \,/\, X[\,3\,|\,3\,]\,) \oplus X[\,3\,|\,3\,] \;=\; \begin{pmatrix} 1/3 & -1/3 & \\ 2/3 & 1/3 & \\ & & 3 \end{pmatrix} & & (\, X \,/\, X[2,3\,|\,2,3]\,) \oplus X[2,3\,|\,2,3] \;=\; \begin{pmatrix} 1 & & \\ & 5 & 7 \\ & 2 & 3 \end{pmatrix} \\
X[\,1\,|\,1\,] \oplus (\, X[\,1\,|\,1\,] \setminus X\,) \;=\; \begin{pmatrix} 2 & & \\ & 1/2 & -1/2 \\ & 1/2 & 1/2 \end{pmatrix} & & X[1,2\,|\,1,2] \oplus (\, X[1,2\,|\,1,2] \setminus X\,) \;=\; \begin{pmatrix} 2 & 3 & \\ 3 & 5 & \\ & & 1 \end{pmatrix}.
\end{array}$$

This diagonalization process can be repeated on submatrices, until the results are diagonal. Figure 2 diagrams the set of block unit diagonalizations for the matrix X, showing how they are derived. Notice each member of the set has the same determinant. Also the diagonal matrix D in the LDU decomposition of X is the fifth matrix on the bottom row.

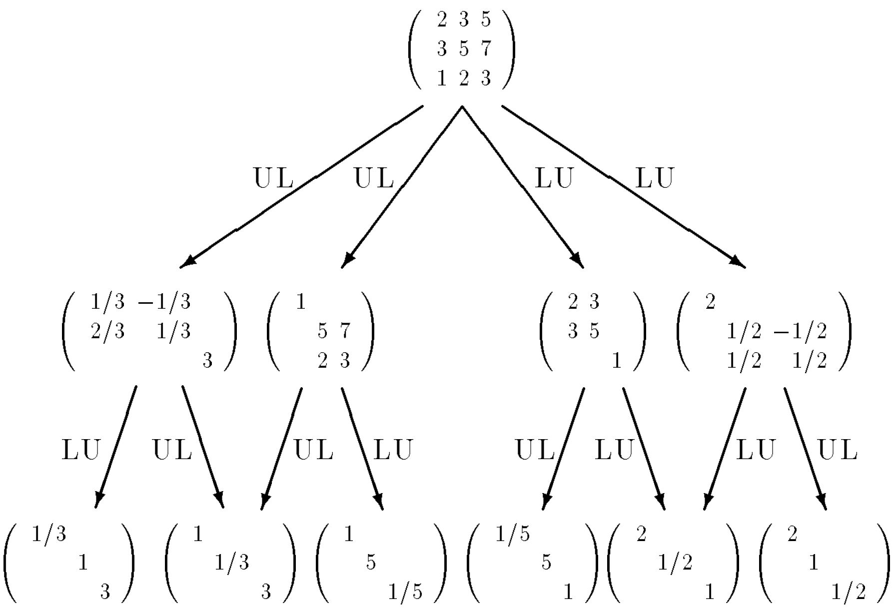


Figure 2: zULU diagonalizations of a matrix

These recursive block unit diagonalizations ultimately yield a (recursive) block LU- or UL decomposition. Although traditionally one uses either LU- or UL-decompositions exclusively, here they can be interleaved. For example:

$$\begin{aligned}
\begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 7 \\ 1 & 2 & 3 \end{pmatrix} &\overset{LU}{=} \begin{pmatrix} 1 & & \\ & 1 & \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & \\ 3 & 5 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 4 \\ & 1 & -1 \\ & & 1 \end{pmatrix} \\
&\overset{UL}{=} \begin{pmatrix} 1 & & \\ & 1 & \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3/5 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/5 & & \\ & 5 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 3/5 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 4 \\ & 1 & -1 \\ & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3/5 & \\ & 1 & \\ -1 & 2/5 & 1 \end{pmatrix} \begin{pmatrix} 1/5 & & \\ & 5 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 4 \\ 3/5 & 1 & 7/5 \\ & & 1 \end{pmatrix}.
\end{aligned}$$

The final decomposition VDW here is very like an LDU- or UDL-decomposition, but instead of V and W having a triangular pattern, they have what we could call a 'zigzag triangular' pattern.

Definition 8 *A* **zigzag UL/LU (zULU) diagonalization** Y *of a nondegenerate matrix* X *is a matrix that can be obtained from* X *in zero or more block unit UL or LU diagonalization steps.*

We say a square nondegenerate matrix X *determines a* **zULU class** $[X]$*, defined by*

$$[X] \quad = \quad \{\ x \mid x \textit{ is a zULU diagonalization of } X\ \}.$$

Specifically, $X \in [X]$, and the diagonal matrix D in the LDU decomposition of X is also in $[X]$. Figure 2 shows the entire set $[X]$ of eleven zULU diagonalizations for the matrix X above.

Lemma 3 *Every member of a zULU class* $[X]$ *has determinant* $\det X$*.*

Proof
LU- and UL-diagonalizations always leave the determinant undisturbed. □

5 zULU Classes obey Categorial Grammar

Both leading and trailing Schur complements define natural quotient operators on zULU classes, and matrix direct sums define a natural product operator.

Formally, let M be the semigroup of matrices under the (associative) multiplicative operation '$\oplus$'. Then three operators can be defined on zULU classes $[X]$, $[Y]$:

$$\begin{aligned}
[X] \oplus [Y] &= \{\ x \oplus y \in M \mid x \in [X],\ y \in [Y]\ \} \\
[X] \,/\, [Y] &= \{\ z \in M \mid \forall y \in [Y],\ z \oplus y \in [X]\ \} \\
[Y] \setminus [X] &= \{\ z \in M \mid \forall y \in [Y],\ y \oplus z \in [X]\ \}.
\end{aligned}$$

For example, If X is as in the previous example, so $X \trianglerighteq (3)$, then $[X]\,/\,[(3)]$ is the class

$$[X] \,/\, [(3)] \quad = \quad \left\{ \begin{pmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{pmatrix},\quad \begin{pmatrix} 1 & \\ & 1/3 \end{pmatrix},\quad \begin{pmatrix} 1/3 & \\ & 1 \end{pmatrix} \right\}.$$

Notice that $[X]\,/\,[(3)] = [X/(3)]$.

Theorem 2 *zULU classes obey the laws of Categorial Grammar.*

That is: if X_1 *and* X_2 *are Categorial Grammar expressions such that* $X_1 \Rightarrow X_2$*, then* X_1 *and* X_2 *are matrix expressions involving Schur complements and direct sum such that* $[X_1] \subseteq [X_2]$.

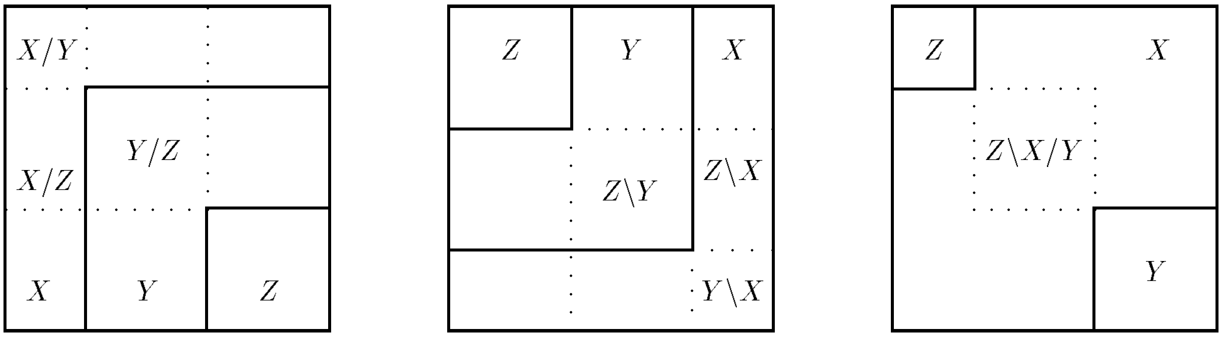
Proof
First, zULU classes obey the axioms, where '1' is the empty matrix (the identity under direct sum) and '$\Rightarrow$' is '$\subseteq$'. The axioms are easy to establish: $([X] \oplus 1) = [X]$, $(1 \oplus [X]) = [X]$, and $(([X] \oplus [Y]) \oplus [Z]) = ([X] \oplus ([Y] \oplus [Z]))$.

Second, zULU classes obey the inference rules as well, since they are easily verified to satisfy the following properties:

$$\begin{aligned}
[X] \oplus [Y] \subseteq [Z] \quad &\text{iff} \quad [X] \subseteq [Z] \,/\, [Y]. \\
[Y] \oplus [X] \subseteq [Z] \quad &\text{iff} \quad [X] \subseteq [Y] \setminus [Z].
\end{aligned}$$

Actually Lambek [13] points out that arbitrary subsets of semigroups will satisfy these properties, when the operators are defined as above. □

A simple pictorial summary of Theorem 2 is possible. Figure 3 portrays various properties of zULU classes, where Y and Z denote specific block submatrices of X, and '$X_1 \Rightarrow X_2$' means '$[X_1] \subseteq [X_2]$'.



$$(X/Y) \cdot (Y/Z) \Rightarrow (X/Z)$$
$$(X/Y) \Rightarrow (X/Z) \,/\, (Y/Z)$$
$$(Y/Z) \Rightarrow (X/Y) \setminus (X/Z)$$
$$X \trianglerighteq Y \trianglerighteq Z$$

$$(Z\backslash Y) \cdot (Y\backslash X) \Rightarrow (Z\backslash X)$$
$$(Y\backslash X) \Rightarrow (Z\backslash Y) \setminus (Z\backslash X)$$
$$(Z\backslash Y) \Rightarrow (Z\backslash X) \,/\, (Y\backslash X)$$
$$Z \trianglelefteq Y \trianglelefteq X$$

$$(Z\backslash X) \,/\, Y \Rightarrow Z \setminus (X/Y)$$
$$Z \trianglelefteq X, \quad X \trianglerighteq Y$$

Figure 3: Lambek's Categorial Grammar relations, portrayed as properties of zULU classes

6 Schur Complements obey Categorial Grammar

If '$\Rightarrow$' has interpretation 'have equal determinants', Schur complements obey Categorial Grammar.

Definition 9 *A matrix expression involving Schur complements and direct sum is* **well-formed** *if in every subexpression* $(\ X \ / \ Y\ )$ *both* X *and* Y *denote matrices such that* $X \trianglerighteq Y$*, and in every subexpression* $(\ Y \setminus X\ )$ *both* X *and* Y *denote matrices such that* $Y \trianglelefteq X$*.*

Theorem 3 *Schur complements obey the laws of Categorial Grammar.*

That is: if X_1 *and* X_2 *are well-formed Categorial Grammar expressions such that* $X_1 \Rightarrow X_2$*, then* X_1 *and* X_2 *are matrix expressions involving Schur complements and direct sum such that* $det\ X_1 = det\ X_2$*.*

Proof

If the matrix expressions X_1 and X_2 are well-formed, then they denote specific matrices A_1 and A_2. We claim that $A_1 \in [X_1]$ and $A_2 \in [X_2]$. This can be proven by inductively establishing (on the size of the expression) each of the following relations:

$$\begin{aligned} [X \oplus Y] &= [X] \oplus [Y] \\ [X \ / \ Y] &\subseteq [X] \,/\, [Y] \\ [Y \setminus X] &\subseteq [Y] \setminus [X]. \end{aligned}$$

For example in the base case where A is the value of $(\ X \ / \ Y\ )$ where X and Y are matrices for which $X \trianglerighteq Y$ (so $(\ X \ / \ Y\ )$ is well-formed), then because $[X] \,/\, [Y] = \{\, z \in M \mid \forall y \in [Y],\ z \oplus y \in [X] \,\}$ specifically $A \in [X] \,/\, [Y]$, and thus $[A] \subseteq [X] \,/\, [Y]$, i.e., $[X \,/\, Y] \subseteq [X] \,/\, [Y]$.

Therefore $[A_1] \subseteq [X_1]$ and $[A_2] \subseteq [X_2]$, so in particular $A_1 \in [X_1]$ and $A_2 \in [X_2]$ as claimed. By Lemma 3, every element of $[X_1]$ (resp. $[X_2]$) shares the common determinant value $det\ X_1 = det\ A_1$ (resp. $det\ X_2 = det\ A_2$). But by Theorem 2, $[X_1] \subseteq [X_2]$, so $det\ X_1 = det\ X_2$. $\square$

It is natural to hope we could strengthen Theorem 3 to directly relate Categorial Grammar with Schur complement *matrices*, rather than Schur complement *determinants*. This hope is encouraged by the example above, in which $[X]/[(3)] = [X/(3)]$. Moreover, we have identities like the quotient property (Lemma 2) which precisely match quotient reduction laws of categorial grammar.

Unfortunately Theorem 3 is no longer valid if we omit the '*det*'s from its statement. In particular, although it is true that

$$det\ (\ X\ /\ (\ A \setminus X\ )\ ) \quad = \quad det\ A,$$

in general Schur complements do not generally satisfy the type raising law, i.e.,

$$(\ X\ /\ (\ A \setminus X\ )\ ) \quad \neq \quad A.$$

For example, with the matrices $X = \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 1 \end{array}\right)$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$(\ X\ /\ (\ A \setminus X\ )\ ) \quad = \quad (\ X\ /\ (1)\ ) \quad = \quad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \neq \quad A.$$

The basic problem is that a matrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has both the UL-diagonalization $(\ X\ /\ D\ ) \oplus D$ and the LU-diagonalization $A \oplus (\ A \setminus X\ )$, but the inference rules

$$\begin{array}{llll} \text{if} & (Y \oplus Z) \Rightarrow X & \text{then} & Y \Rightarrow (X/Z) \\ \text{if} & Y \Rightarrow (X/Z) & \text{then} & (Y \oplus Z) \Rightarrow X \end{array}$$

where '$\Rightarrow$' means 'is a zULU diagonalization of' are valid only for the UL-diagonalization. In particular, although

'$A \oplus (\ A \setminus X\ )$ is a zULU diagonalization of X' is true

generally

'A is a zULU diagonalization of $(\ X\ /\ (\ A \setminus X\ )\ )$' is false.

However, consider the following fragment of the Categorial Grammar theory intended for use with Schur complements, where '$\Rightarrow$' means 'is an LU diagonalization of':

Axioms	$(X \oplus 1) \Rightarrow X$			
	$X \Rightarrow (X \oplus 1)$			
	$(1 \oplus X) \Rightarrow X$			
	$X \Rightarrow (1 \oplus X)$			
	$((X \oplus Y) \oplus Z) \Rightarrow (X \oplus (Y \oplus Z))$			
	$(X \oplus (Y \oplus Z)) \Rightarrow ((X \oplus Y) \oplus Z)$			
	$(X/X) \Rightarrow 1$			
	$((X/Y) \oplus Y) \Rightarrow X$ (provided $X \trianglerighteq Y$)			
Rules of Inference	if $X \Rightarrow Y$ and $Y \Rightarrow Z$	then	$X \Rightarrow Z$	
	if $(Y \oplus Z) \Rightarrow X$	then	$Y \Rightarrow (X/Z)$	(provided $X \trianglerighteq Z$)
	if $Y \Rightarrow (X/Z)$	then	$(Y \oplus Z) \Rightarrow X$	(provided $X \trianglerighteq Z$)
	if $Y \Rightarrow Z$	then	$(X \oplus Y) \Rightarrow (X \oplus Z)$	
	if $Y \Rightarrow Z$	then	$(Y \oplus X) \Rightarrow (Z \oplus X)$	

By design, these rules address only LU-diagonalizations, and cannot produce the confusion above.

Theorem 4 *Schur complements obey the axioms and inference rules above (a fragment of the Categorial Grammar theory).*

Specifically: if X_1 and X_2 are expressions such that $X_1 \Rightarrow X_2$ with the axioms and inference rules above, then X_1 and X_2 are matrix expressions involving Schur complements and direct sum, and X_1 is an LU diagonalization of X_2. Furthermore, if X_1 does not involve direct sum, then $X_1 = X_2$.

Proof

The first statement follows since each axiom and inference rule holds of LU diagonalizations. When X_1 does not involve direct sum, it denotes a full matrix (in general). Since X_1 is a diagonalization of X_2, it must then also be the case that X_2 is full, and $X_1 = X_2$. □

For example, with these axioms and inference rules we can derive the Schur Complement Quotient law (Lemma 2)

$$((A\ /\ C)\ /\ (B\ /\ C)) \quad = \quad (A\ /\ B) \qquad (\text{provided } A \trianglerighteq B, \quad B \trianglerighteq C)$$

in the following way, without resorting to complex block matrix formulas:

1.	$(B/C) \oplus C \Rightarrow B$	axiom: $X/Y \oplus Y \Rightarrow X$ (provided $B \trianglerighteq C$)
2.	$(A/B) \oplus ((B/C) \oplus C) \Rightarrow (A/B) \oplus B$	1. & rule: if $Y \Rightarrow Z$ then $X \oplus Y \Rightarrow X \oplus Z$
3.	$(A/B) \oplus B \Rightarrow A$	axiom: $X/Y \oplus Y \Rightarrow X$ (provided $A \trianglerighteq B$)
4.	$(A/B) \oplus ((B/C) \oplus C) \Rightarrow A$	2. & 3. & rule: if $X \Rightarrow Y$ and $Y \Rightarrow Z$ then $X \Rightarrow Z$
5.	$((A/B) \oplus (B/C)) \oplus C \Rightarrow (A/B) \oplus ((B/C) \oplus C))$	axiom: $((X \oplus Y) \oplus Z) \Rightarrow (X \oplus (Y \oplus Z))$
6.	$((A/B) \oplus (B/C)) \oplus C \Rightarrow A$	5. & 4. & rule: if $X \Rightarrow Y$ and $Y \Rightarrow Z$ then $X \Rightarrow Z$
7.	$((A/B) \oplus (B/C)) \Rightarrow (A/C)$	6. & rule: if $Y \oplus Z \Rightarrow X$ then $Y \Rightarrow (X/Z)$
8.	$(A/B) \Rightarrow ((A/C)/(B/C))$	7. & rule: if $Y \oplus Z \Rightarrow X$ then $Y \Rightarrow (X/Z)$.

The final consequent

$$(A\ /\ B) \quad \text{is an LU diagonalization of} \quad ((A\ /\ C)\ /\ (B\ /\ C))$$

actually yields the quotient law equation since the left side does not involve direct sum. Notice also that the seventh step of the derivation above gives the composition law

$$(A\,/\,B) \ \oplus \ (B\,/\,C) \quad \text{is an LU diagonalization of} \quad (A\,/\,C)$$

which may be a 'new' property of Schur Complements.

Many other properties follow from these axioms and inference rules. For example

$$(A \ \oplus \ (B\ /\ C)) \quad \text{is an LU diagonalization of} \quad ((A \ \oplus \ B)\ /\ C) \qquad (\text{provided } B \trianglerighteq C)$$

is derivable in the following way:

1.	$(B/C) \oplus C \Rightarrow B$	axiom: $X/Y \oplus Y \Rightarrow X$ (provided $B \trianglerighteq C$)
2.	$A \oplus ((B/C) \oplus C) \Rightarrow A \oplus B$	1. & rule: if $Y \Rightarrow Z$ then $X \oplus Y \Rightarrow X \oplus Z$
3.	$(A \oplus (B/C)) \oplus C \Rightarrow A \oplus ((B/C) \oplus C)$	axiom: $((X \oplus Y) \oplus Z) \Rightarrow (X \oplus (Y \oplus Z))$
4.	$(A \oplus (B/C)) \oplus C \Rightarrow A \oplus B$	3. & 2. & rule: if $X \Rightarrow Y$ and $Y \Rightarrow Z$ then $X \Rightarrow Z$
5.	$A \oplus (B/C) \Rightarrow (A \oplus B)/C$	4. & rule: if $Y \oplus Z \Rightarrow X$ then $Y \Rightarrow (X/Z)$.

Studying this property suggests that the rules are not yet complete: the property should yield an equation, yet the converse does not appear to be derivable. Still, the converse holds only provided that $B \trianglerighteq C$.

7 Concluding Remarks

This paper has shown properties satisfied by Schur complements can be derived using Categorial Grammar, and that there is a natural connection between the two concepts. Categorial Grammar thus gives new perspectives on LU decomposition and Schur complements. For example, Figure 3 suggests various identities satisfied by Schur complements, and Categorial Grammar makes it possible to derive the quotient property of Schur complements without complex matrix manipulations. It should be possible to extend the connection in various ways, including the following possibilities.

7.1 Applications in Gaussian Elimination

The Schur complement has gradually gained in recognition as a useful tool in various aspects of matrix analysis. The results collected here help summarize why, and provide evidence that this trend should continue.

Our work suggests an interesting avenue for research: have Gaussian elimination generate zULU decompositions, rather than LU decompositions. That is, let a **zigzag UL/LU (zULU) decomposition** $X = VYW$ of a matrix X give a diagonal matrix Y and a pair of matrices V and W resulting from recursive block unit LU- or UL-diagonalization. The increased flexibility of being able to select either LU or UL decomposition dynamically should permit fast matrix decomposition with better roundoff properties than with just LU decomposition.

7.2 Future Work in Linear Algebra and Matroid Theory

The results above hint at more fundamental connections between general Categorial Grammar and Schur complements. In particular:

- *Duals, Adjoints, and Inverses*

  We can introduce a dual operator on matrices defined by

$$X^R = RXR$$

  where $R = (r_{i,j})$ is the reversal permutation defined by $r_{i,j} = 1$ if $i = (n - j + 1)$, 0 otherwise. The dual X^R is just X with rows and columns reversed, so that

$$\begin{aligned} (X \oplus Y)^R &= (Y^R \oplus X^R) \\ (X \,/\, Y)^R &= (Y^R \setminus X^R) \\ (X \setminus Y)^R &= (Y^R \,/\, X^R). \end{aligned}$$

  Perhaps this duality can be exploited to obtain interesting properties of the quotient structure (since reversal has an analogue in formal models of parsing).

  Also there are several matrix adjoints '$\star$' that have the following properties:

$$\begin{aligned} (X \oplus Y)^\star &= (X^\star \oplus Y^\star) \\ (X \,/\, Y)^\star &= (X^\star \,/\, Y^\star) \\ (X \setminus Y)^\star &= (X^\star \setminus Y^\star). \end{aligned}$$

  For example, these identities hold when '$\star$' is either matrix transpose or Hermitian adjoint (conjugate transpose).

  Finally various properties also hold for matrix inverse:

$$\begin{aligned} (X \oplus Y)^{-1} &= (X^{-1} \oplus Y^{-1}) \\ (X \,/\, Y)^{-1} \setminus X^{-1} &= Y^{-1} \\ X^{-1} \,/\, (Y \setminus X)^{-1} &= Y^{-1}. \end{aligned}$$

These are verified by noting that when $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and A and D are invertible,

$$\begin{aligned} X^{-1} &= \begin{pmatrix} A^{-1}+A^{-1}B(A\backslash X)^{-1}CA^{-1} & -A^{-1}B(A\backslash X)^{-1} \\ -(A\backslash X)^{-1}CA^{-1} & (A\backslash X)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (X/D)^{-1} & -(X/D)^{-1}BD^{-1} \\ -D^{-1}C(X/D)^{-1} & D^{-1}+D^{-1}C(X/D)^{-1}BD^{-1} \end{pmatrix}. \end{aligned}$$

Recent work in Lambek systems has considered the use of unary operators. For example, Moortgat [15] introduces modal operators with the following new rules:

$$\boxed{\begin{array}{llll} \text{if} & \Diamond X \Rightarrow Y & \text{then} & X \Rightarrow \Box Y \\ \text{if} & X \Rightarrow \Box Y & \text{then} & \Diamond X \Rightarrow Y. \end{array}}$$

In addition, he considers the effects of including a variety of modal axioms, including among others the distributivity postulates

$$\boxed{\begin{array}{lcl} \Box(X/Y) & \Rightarrow & (\Box X)/(\Box Y) \\ \Diamond(X\cdot Y) & \Rightarrow & (\Diamond X)\cdot(\Diamond Y) \end{array}}$$

resembling the adjoint identities above.

- *General Schur Complement*
  Generally, for any nonsingular principal submatrix $X[\,\alpha\,|\,\alpha\,]$ where α is any subsequence of $1,\ldots,n$, if α' is its sequence complement in $1,\ldots,n$, one defines **the Schur complement of $X[\,\alpha\,|\,\alpha\,]$ in X** as

  $$(\ X\ /\ X[\,\alpha\,|\,\alpha\,]\ ) \quad = \quad \Big(\ X[\,\alpha'\,|\,\alpha'\,] \quad - \quad X[\,\alpha'\,|\,\alpha\,]\cdot(X[\,\alpha\,|\,\alpha\,])^{-1}\cdot X[\,\alpha\,|\,\alpha'\,]\ \Big).$$

  For example, Schur's identity for determinants then becomes:

  $$\frac{det\ X[\,\alpha,\beta\,|\,\alpha,\beta\,]}{det\ X[\,\beta\,|\,\beta\,]} \quad = \quad det\ (\ X[\,\alpha,\beta\,|\,\alpha,\beta\,]\ /\ X[\,\beta\,|\,\beta\,]\ ).$$

  In this paper we have focused on Schur complements with respect to leading and trailing principal submatrices, but this equation shows that they could be generalized to permit arbitrary principal submatrices. An equivalent generalization is to allow identical permutation of rows and columns of X, so that the matrix can be reordered to place any desired principal submatrix in the upper left corner.

  Incorporating permutation has also been considered in Categorial Grammar. In fact, permutation is sometimes incorporated by adding the equivalence

  $$(X\ /\ Y) \quad \equiv \quad (Y \setminus X).$$

  Refinements of this simple approach have been proposed [10, 14], motivated by better modeling of natural language than is possible with Categorial Grammar without permutation.

- *Minors and Matroids*
  Generalizations of both the preceding ideas have been studied in the framework of matroids. Specifically, Oxley ([18], ch.3) reviews related results of Tutte [24] on 'minors' of matroids. Any matrix X defines a matroid M of linearly independent sets of columns of X, and the

complements of these sets define the dual matroid M^*. If T is a set of columns of X, the natural quotients satisfy the identity

$$M \,/\, T \quad = \quad (\, M^* \setminus T\,)^*,$$

and 'minors' of the matroids can be defined that capitalize on this duality. It might be interesting to reconcile the work in this paper with that on matroids.

Acknowledgement

I am indebted to Michael Moortgat for pointing out an error in an earlier version of this paper, and for giving me an overview of some recent work on categorial grammar. In particular he pointed out generalizations of categorial grammar involving unary operators.

References

[1] J. van Benthem, "The Lambek Calculus", in [16], 35–68, 1988.

[2] D. Carlson, E. Haynsworth, T. Markham, "A generalization of the Schur complement by means of the Moore-Penrose inverse", *SIAM J. Appl. Math.* **8**:1, 169–175, 1974.

[3] R.W. Cottle, "Manifestations of the Schur Complement", *Linear Algebra and its Applications* **8**, 189–211, 1974.

[4] D.E. Crabtree and E.V. Haynsworth, "An Identity for the Schur Complement of a Matrix", *Proc. AMS* **22**, 364–366 (1969).

[5] F.R. Gantmacher, *The Theory of Matrices, Volume One.* NY: Chelsea Publishing Co, 1959.

[6] E. Haynsworth, "Determination of the inertia of a partitioned Hermitian matrix", *Linear Algebra and Appl.* **1**, 73–81, 1968.

[7] E. Haynsworth, "Applications of an inequality for the Schur complement", *Proc. Amer. Math. Soc.* **24**, 512–516, 1970.

[8] E. Haynsworth, "Reduction of a matrix using properties of the Schur complement", *Linear Algebra and Appl.* **3**, 23–29, 1970.

[9] R.A. Horn, C.R. Johnson, *Matrix Analysis*, NY: Cambridge University Press, 1992.

[10] E.L. Keenan, A. Timberlake, "Natural Language Motivations for Extending Categorial Grammar", in [16], 265–295, 1988.

[11] J. Lambek, "The Mathematics of Sentence Structure", *American Mathematical Monthly* **65**:1, 154–170, 1958.

[12] J. Lambek, *Rings and Modules*, NY: Chelsea, 1966.

[13] J. Lambek, "Categorial and Categorical Grammars", in [16], 297–317, 1988.

[14] M. Moortgat, *Categorial Investigations: Logical and Linguistic Aspects of the Lambek Calculus*, Providence, RI: Foris Publications, 1988.

[15] M. Moortgat, "Multimodal linguistic inference", *Journal of Logic, Language and Speech (OTS)*, to appear. Available from `moortgat@let.ruu.nl`.

[16] R.T. Oehrle, E. Bach, D. Wheeler (eds.), *Categorial Grammars and Natural Language Structures*, Boston: D. Reidel, 1988.

[17] A. Ostrowski, "A new proof of Haynsworth's quotient formula for Schur complements", *Linear Algebra and Appl.* **4**, 389-392, 1971.

[18] J.G. Oxley, *Matroid Theory*, NY: Oxford University Press, 1992.

[19] D.S. Parker, "Random Butterfly Transformations with applications in Computational Linear Algebra", Technical Report CSD-950023, UCLA Computer Science Department, 1995.

[20] D.S. Parker, "Explicit Formulas for the Results of Gaussian Elimination", Technical Report CSD-950025, UCLA Computer Science Department, 1995.

[21] D.S. Parker, D. Le, "How to Eliminate Pivoting from Gaussian Elimination — by Randomizing Instead", Technical Report CSD-950022, UCLA Computer Science Department, 1995.

[22] D.S. Parker, D. Lê, "Quadtree Matrix Algorithms Revisited: Basic Issues and their Resolution", Technical Report CSD-950028, UCLA Computer Science Department, 1995.

[23] I. Schur, "Potenzreihen im Innern des Einheitskreises", *J. Reine Angew. Math.*, **147**, 205–232, 1917.

[24] W.T. Tutte, "A Homotopy Theorem for Matroids. I, II", *Trans. AMS* **88**, 144–174, 1958.